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Global Gevrey hypoellipticity for twisted Laplacians

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Abstract In this paper we study global Gevrey/analytic hypoellipticity of the anisotropic twisted Laplacian

$$L_{p,q} = \sum_{j=1}^{n} \Big(D_{x_j} - y_j^{p_j}/2\Big)^2 + \sum_{j=1}^{n} \Big(D_{y_j} + x_j^{q_j}/2\Big)^2,$$

depending on the powers p_j, q_j, $1 \leq j \leq n$, which determine the anisotropy. It turns out that when $p_j = q_j = 1$ for all j (the case of the "classical" twisted Laplacian) then $L_{p,q}$ is globally analytic hypoelliptic, whereas when for at least one j either $p_j > 1$ or $q_j > 1$, the operator $L_{p,q}$ is globally Gevrey $G^{\sigma,\tau}$ hypoelliptic, for suitable multi-indices σ and τ.

Keywords Twisted Laplacian · Global analytic hypoellipticity · Global Gevrey hypoellipticity · Anisotropic Gevrey regularity

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1 Introduction and main results

Let $p_j, q_j \geq 1$, $1 \leq j \leq n$, be integers. In $\mathbb{R}^{2n} = \mathbb{R}^n_x \times \mathbb{R}^n_y$ we consider the *anisotropic* twisted Laplacian (where $D = -i\partial$)

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$$L_{p,q} = \sum_{j=1}^{n} \left(D_{x_j} - y_j^{p_j}/2 \right)^2 + \sum_{j=1}^{n} \left(D_{y_j} + x_j^{q_j}/2 \right)^2 . \qquad (1.1)$$

When $p_j = q_j = 1$ for all $j = 1, \ldots, n$, we shall simply write L for the twisted Laplacian (which is then the *isotropic*, or *classical*, twisted Laplacian).

The importance of the twisted Laplacian L is well-known, in that not only is it a model of Schrödinger operator with a constant magnetic field, but it also describes the action of the "reduced" Heisenberg group $\mathscr{H}_n = \mathbb{R}^n \times \mathbb{R}^n \times (\mathbb{R}/2\pi\mathbb{Z})$, and is connected to Hermite operators in $\mathbb{R}^{2n}$ and to the Kohn sub-Laplacian on the Heisenberg group. Because of this, it has recently received a good deal of interest, as far as its spectral and hypoellipticity properties are concerned; see for instance [4,6,7,9,10] and references listed there.

In this paper, we study *global* hypoellipticity properties of the more general (*anisotropic*) twisted Laplacian $L_{p,q}$. The operator $L_{p,q}$ is related to a magnetic Schrödinger operator as follows. One considers the magnetic vector potential $A(x, y)$, represented as a 1-form ω_A,

$$\omega_A = \frac{1}{2} \sum_{j=1}^{n} \left(y_j^{p_j} dx_j - x_j^{q_j} dy_j \right),$$

which gives rise to the magnetic field $B(x, y)$, represented as a 2-form σ_B,

$$\sigma_B = d\omega_A = -\frac{1}{2} \sum_{j=1}^{n} \left(p_j y_j^{p_j - 1} + q_j x_j^{q_j - 1} \right) dx_j \wedge dy_j ,$$

so that the associated magnetic Schrödinger operator is indeed given by the twisted Laplacian $L_{p,q}$.

Note that $L_{p,q}$ is locally elliptic, but *not* globally elliptic. Hence its local regularity properties are well-known while on the global side things are not as well-known. For the isotropic twisted Laplacian (i.e. $p_j = q_j = 1$), there has been work concerned with the spectral and global properties, for example the hypoellipticity in $\mathscr{S}(\mathbb{R}^{2n})$ and in the Gelfand-Shilov spaces $\mathcal{S}_\nu^\mu(\mathbb{R}^{2n})$ (see Dasgupta and Wong [2]), the spectrum and fundamental solutions (see [2,8]). The corresponding properties for the anisotropic case are as yet not so well-explored. Here we focus on global hypoellipticity in the Gevrey and analytic category (see [1,5] for detailed expositions on global hypoellipticity and Gevrey classes). We shall see that while L is globally analytic hypoelliptic, when for some j for the anisotropy indices p and q one has either $p_j > 1$ or $q_j > 1$, the twisted Laplacian $L_{p,q}$ is globally Gevrey hypoelliptic on suitable anisotropic Gevrey spaces (see Definition 1.1 below). The proof of our result in the case of global Gevrey hypoellipticity, goes in two steps:

- In the first step, by using an auxiliary parametric family attached to $L_{p,q}$, we prove a global hypoellipticity result in the Sobolev spaces $H^m(\mathbb{R}^{2n})$;
- In the second one, we obtain the sought a priori inequalities for the L^2 norms of the derivatives, which are crucial for the analytic/Gevrey hypoellipticity.

In the final section of the paper we shall give a *direct* proof of the global analytic hypoellipticity of L, by exploiting the explicit formula of the fundamental solution of L.

Throughout the proofs, for notational simplicity and without loss of generality, we shall assume $n = 1$.

We next recall the anisotropic Gevrey spaces on $\mathbb{R}^d$ that we denote by $G^\sigma = G^{\sigma_1,\dots,\sigma_d}$, $\sigma = (\sigma_1, \dots, \sigma_d)$.

Definition 1.1 We say that a differential operator P is globally $G^{\sigma_1,\sigma_2,\dots,\sigma_d}$-hypoelliptic in $\mathbb{R}^d$, if for any given $f \in C^\infty(\mathbb{R}^d)$ satisfying

$$\forall\, \alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}_+^d, \quad \left\| D^\alpha f \right\|_{L^2} \le M_1^{|\alpha|+1} \prod_{j=1}^d \left(\alpha_j!\right)^{\sigma_j} \tag{1.2}$$

for some constant $M_1 > 0$, then

$$u \in L^2(\mathbb{R}^d) \ \text{ and } \ Pu = f \Longrightarrow \forall\, \alpha \in \mathbb{Z}_+^d, \quad \left\| D^\alpha u \right\|_{L^2} \le M_2^{|\alpha|+1} \prod_{j=1}^d \left(\alpha_j!\right)^{\sigma_j}.$$

for another constant $M_2 > 0$.

Note that the $G^{1,1,\dots,1}$-hypoellipticity is just the global analytic hypoellipticity in the whole space $\mathbb{R}^d$. In our case, we shall have $d = 2n$.

We next state the main results of this paper.

Our first result is an H^∞-hypoellipticity result, which is basic for the subsequent global Gevrey-hypoellipticity result. Recall that $H^\infty = \bigcap_{m\ge 0} H^m$ with $H^m = H^m(\mathbb{R}^{2n})$ the usual Sobolev space.

Theorem 1.2 *Let $m \in \mathbb{Z}_+$ be given. Then the twisted laplacian $L_{p,q}$ is globally H^m-hypoelliptic in $\mathbb{R}^{2n}$, in the sense that*

$$\begin{aligned} &u \in L^2, L_{p,q} u \in H^m \\ \Longrightarrow\ & u \in H^m,\ \left(D_{x_j} - y_j^{p_j}/2\right) D^\alpha u,\ \left(D_{y_j} + x_j^{q_j}/2\right) D^\alpha u \in L^2,\ |\alpha| \le m,\ 1 \le j \le n. \end{aligned}$$

Our second result is the actual global Gevrey-hypoellipticity of $L_{p,q}$.

Theorem 1.3 *The twisted laplacian $L_{p,q}$ is globally $G^{\sigma_1,\sigma_2,\dots\sigma_n,\tau_1,\tau_2\dots,\tau_n}$-hypoelliptic in $\mathbb{R}^{2n}$, where (σ_j, τ_j), $j = 1, 2, \dots, n$, are defined as follows:*

$$\begin{cases} (\sigma_j,\ \tau_j) = \left((p_j+1)/2,\ (q_j+1)/2\right), & p_j = 1,\ \textit{or}\ q_j = 1; \\ \sigma_j = \tau_j = \max\left\{(p_j+1)/2,\ (q_j+1)/2\right\}, & p_j, q_j \ge 2, \textit{both } p_j \textit{ and } q_j \textit{ are odd}; \\ \sigma_j = \tau_j = \max\left\{(2p_j+2)/3,\ (2q_j+2)/3\right\}, & \textit{otherwise}. \end{cases}$$

In particular, the isotropic twisted Laplacian L is globally analytic hypoelliptic in $\mathbb{R}^{2n}$.

As already mentioned, we shall actually prove Theorems 1.2 and 1.3 in the case $n = 1$, for the sake of notational simplicity.

2 Twisted Laplacian with parameters

We begin with fixing some notation used throughout this paper. Given $\lambda \geq 1$, denote by $Z_\lambda = (Z_{1,\lambda}, Z_{2,\lambda})$ the "twisted gradient" defined by the first-order operators $Z_{j,\lambda}$, $j = 1, 2$, given by

$$Z_{1,\lambda} = D_x - \frac{\lambda^{p+1}}{2} y^p, \quad Z_{2,\lambda} = D_y + \frac{\lambda^{q+1}}{2} x^q.$$

Note that the $Z_{j,\lambda}$, $j = 1, 2$, are (formally) self-adjoint operators in $L^2(\mathbb{R}^2)$, with common core $\mathscr{S}(\mathbb{R}^2)$. Then the Baker-Campbell-Hausdorff formula shows that (see Nier [3], Lemma 4.14)

$$\forall\, v \in \mathscr{S}(\mathbb{R}^2),\ \forall\, I = (i_1, \ldots, i_k) \in \{1, 2\}^k,$$
$$\left\| |Z_{I,\lambda}|^{1/|I|} v \right\|_{L^2} \leq C_* \left(\sum_{j=1}^{2} \|Z_{j,\lambda} v\|_{L^2} + \|v\|_{L^2} \right),$$

where C_* is some constant independent of λ, $|I|$ = length of the commutator $Z_{I,\lambda}$ given by

$$Z_{I,\lambda} = [Z_{i_1,\lambda},\ [Z_{i_2,\lambda}, \ldots, [Z_{i_{k-1},\lambda},\ Z_{i_k,\lambda}]\,]\,].$$

In particular, we have that

$$\forall\, v \in \mathscr{S}(\mathbb{R}^2),\ \forall\, \lambda \geq 1,$$
$$\|\lambda v\|_{L^2} + \left\| |\lambda^{p+1} y^{p-1} + \lambda^{q+1} x^{q-1}|^{1/2} u \right\|_{L^2} \leq C_* \left(\sum_{j=1}^{2} \|Z_{j,\lambda} v\|_{L^2} + \|v\|_{L^2} \right),$$

and when $p, q \geq 2$ we have, for any given $v \in \mathscr{S}(\mathbb{R}^2)$ and any given $\lambda \geq 1$,

$$\left\|\lambda^{(p+1)/3} y^{(p-2)/3} v\right\|_{L^2} + \left\|\lambda^{(q+1)/3} x^{(q-2)/3} v\right\|_{L^2} \leq C_* \left(\sum_{j=1}^{2} \|Z_{j,\lambda} u\|_{L^2} + \|v\|_{L^2} \right).$$

As a result, we can find two positive constants C and $\lambda_0 \geq 1$, both depending only on the above C_*, such that

$$\forall\, v \in \mathscr{S}(\mathbb{R}^2),\ \forall\, \lambda \geq \lambda_0,$$
$$\left\| |\lambda^{p+1} y^{p-1} + \lambda^{q+1} x^{q-1}|^{1/2} v \right\|_{L^2} + \left\| \lambda v \right\|_{L^2} \leq C \sum_{j=1}^{2} \left\| Z_{j,\lambda} v \right\|_{L^2}, \tag{2.1}$$

and when $p, q \geq 2$,

$$\forall\, v \in \mathscr{S}(\mathbb{R}^2),\ \forall\, \lambda \geq \lambda_0, \quad \left\| \lambda^{\frac{p+1}{3}} y^{\frac{p-2}{3}} v \right\|_{L^2} + \left\| \lambda^{\frac{q+1}{3}} x^{\frac{q-2}{3}} v \right\|_{L^2} \leq C \sum_{j=1}^{2} \left\| Z_{j,\lambda} v \right\|_{L^2}. \tag{2.2}$$

From now on, we let $\lambda \geq \lambda_0$ be fixed such that the above two inequalities (2.1) *and* (2.2) *are fulfilled.*

We define the twisted Laplacian with parameter λ as follows:

$$L_{p,q;\lambda} = \sum_{j=1}^{2} Z_{j,\lambda}^2 = \left(D_x - \frac{\lambda^{p+1}}{2} y^p \right)^2 + \left(D_y + \frac{\lambda^{q+1}}{2} x^q \right)^2. \tag{2.3}$$

Next, given $k \in \mathbb{N}$, we consider the space

$$\mathcal{H}^k_{Z_\lambda} = \left\{ u \in \mathscr{S}'(\mathbb{R}^2); \quad \forall\, |\alpha| \leq k,\ Z_{j,\lambda} D^\alpha u \in L^2,\ j = 1, 2 \right\}.$$

Note that

$$H^{k+1} \bigcap \mathcal{H}^k_{Z_\lambda} \subset \left\{ u \in \mathscr{S}'(\mathbb{R}^2); \quad \forall\, |\alpha| \leq k,\ \langle x \rangle^q\, D^\alpha u,\ \langle y \rangle^p\, D^\alpha u \in L^2 \right\}$$

(where $\langle x \rangle = (1 + |x|^2)^{1/2}$, and analogously for $\langle y \rangle$). We shall write

$$\mathcal{H}^\infty_{Z_\lambda} := \bigcap_{k \geq 1} \mathcal{H}^k_{Z_\lambda} \quad \text{and} \quad \left\| Z_\lambda g \right\|_{L^2} := \left(\left\| Z_{1,\lambda} g \right\|^2_{L^2} + \left\| Z_{2,\lambda} g \right\|^2_{L^2} \right)^{1/2}.$$

Proposition 2.1 *Let $L_{p,q;\lambda}$ and $L_{p,q}$ be the twisted Laplacians defined above. Then $L_{p,q}$ is globally G^{σ_1,σ_2}-hypoelliptic (resp. H^m-hypoelliptic) in $\mathbb{R}^2$ if $L_{p,q;\lambda}$ is globally G^{σ_1,σ_2}-hypoelliptic (resp. H^m-hypoelliptic) in $\mathbb{R}^2$.*

Proof Let $u \in L^2(\mathbb{R}^2)$ be a solution of $L_{p,q} u = f$ for $f \in C^\infty(\mathbb{R}^2)$. Then, with $f \in G^{\sigma_1,\sigma_2}$, resp. $f \in H^m$, we have

$$L_{p,q;\lambda} u_\lambda = f_\lambda,$$

where

$$u_\lambda(x, y) = u(\lambda x, \lambda y), \;\; f_\lambda(x, y) = \lambda^2 f(\lambda x, \lambda y).$$

Observe that for $\alpha \in \mathbb{Z}_+^2$,

$$\begin{aligned} \left\| D^\alpha u_\lambda \right\|_{L^2} &= \lambda^{-1+|\alpha|} \left\| D^\alpha u \right\|_{L^2}, \quad \left\| D^\alpha f_\lambda \right\|_{L^2} = \lambda^{1+|\alpha|} \left\| D^\alpha f \right\|_{L^2}, \\ \left\| Z_{j,\lambda} D^\alpha u_\lambda \right\|_{L^2} &= \lambda^{|\alpha|} \left\| Z_{j,1} D^\alpha u \right\|_{L^2}, \quad j = 1, 2. \end{aligned}$$

Then a direct verification shows that the global G^σ-hypoellipticity (resp. H^m-hypoellipticity) of $L_{p,q}$ can be deduced from that of $L_{p,q;\lambda}$. The proof is complete. □

3 A key estimate

We shall throughout set $\sum_{j=k}^m = 0$ whenever $m < k$.

Proposition 3.1 *Let $p, q \geq 1$ and let $L_{p,q;\lambda}$ be the operator given in (2.3). Then there exists a constant $C_0 > 0$, depending only on p, q and the constant C given in (2.1), such that, for every integer $m \geq 1$ and any given $v \in H^\infty \bigcap \mathcal{H}^\infty_{Z_\lambda}$, we have:*

(i) If $p, q \geq 2$ then

$$\begin{aligned} &\left\| \lambda D_x^m v \right\|_{L^2} + \left\| \lambda D_y^m v \right\|_{L^2} + \left\| Z_\lambda D_x^m v \right\|_{L^2} + \left\| Z_\lambda D_y^m v \right\|_{L^2} \\ &\leq C_0 \left(\left\| D_x^m L_{p,q;\lambda} v \right\|_{L^2} + \left\| D_y^m L_{p,q;\lambda} v \right\|_{L^2} \right) \\ &\quad + C_0 \lambda^{p+q} \left(m^{A_{p,q}} \left\| Z_\lambda D_x^{m-1} v \right\|_{L^2} + m^{B_{p,q}} \left\| Z_\lambda D_y^{m-1} v \right\|_{L^2} \right) \\ &\quad + C_0 \lambda^{p+q} \left(\frac{m!}{(2q)!(m-2q)!} \left\| \lambda D_x^{m-2q} v \right\|_{L^2} + \frac{m!}{(2p)!(m-2p)!} \left\| \lambda D_y^{m-2p} v \right\|_{L^2} \right) \\ &\quad + C_0 \lambda^{p+q} \left(\sum_{j=2}^{2q-1} \frac{m!}{(m-j)!} \left\| Z_\lambda D_x^{m-j} v \right\|_{L^2} + \sum_{j=2}^{2p-1} \left\| Z_\lambda D_y^{m-j} v \right\|_{L^2} \right) \\ &\quad + C_0 \lambda^{p+q} \left(\sum_{j=2}^{2q-1} \frac{m!}{(m-j)!} \left\| \lambda D_x^{m-j} v \right\|_{L^2} + \sum_{j=2}^{2p-1} \left\| \lambda D_y^{m-j} v \right\|_{L^2} \right) \qquad (3.1) \\ &\quad + C_0 \lambda^{p+q} \left(\sum_{j=2}^{2q-1} \frac{m!}{(m-j)!} \left\| \lambda D_x^{m-j+1} v \right\|_{L^2} + \sum_{j=2}^{2p-1} \frac{m!}{(m-j)!} \left\| \lambda D_y^{m-j+1} v \right\|_{L^2} \right) \\ &\quad + C_0 \lambda^{p+q} \left(\sum_{j=3}^{q} \frac{(m-1)!}{(m-j)!} \left\| Z_\lambda D_x^{m-j+1} v \right\|_{L^2} + \sum_{j=3}^{p} \frac{(m-1)!}{(m-j)!} \left\| Z_\lambda D_y^{m-j+1} v \right\|_{L^2} \right) \\ &\quad + C_0 \lambda^{p+q} \left(\sum_{j=3}^{q} \frac{(m-1)!}{(m-j)!} \left\| \lambda D_x^{m-j+2} v \right\|_{L^2} + \sum_{j=3}^{p} \frac{(m-1)!}{(m-j)!} \left\| \lambda D_y^{m-j+2} v \right\|_{L^2} \right), \end{aligned}$$

where the exponents $A(p,q)$ and $B(p,q)$ in the third line are given by

$$\begin{cases} A(p,q)=\frac{q+1}{2},\ B(p,q)=\frac{p+1}{2}, & \textit{both } p \text{ and } q \text{ are odd},\\ A(p,q)=\frac{2q+2}{3},\ B(p,q)=\frac{2p+2}{3}, & \textit{otherwise}.\end{cases}$$

(ii) If $p\ge 1$ and $q=1$ then

$$\begin{aligned}&\left\|\lambda D_x^m v\right\|_{L^2}+\left\|Z_\lambda D_x^m v\right\|_{L^2}\\ &\quad\le C_0\left\|D_x^m L_{p,q;\lambda}v\right\|_{L^2}+C_0\lambda^q m\left\|\lambda D_x^{m-1}v\right\|_{L^2}+C_0\lambda^{q+1}m^2\left\|\lambda D_x^{m-2}v\right\|_{L^2}\end{aligned}\tag{3.2}$$

and

$$\begin{aligned}&\left\|\lambda D_y^m v\right\|_{L^2}+\left\|Z_\lambda D_y^m v\right\|_{L^2}\\ &\quad\le C_0\left\|D_y^m L_{p,q;\lambda}v\right\|_{L^2}+C_0\left\|\lambda D_x^m v\right\|_{L^2}\\ &\qquad+C_0\lambda^p\left(m^{(p+1)/2}\left\|Z_\lambda D_y^{m-1}v\right\|_{L^2}+\frac{m!}{(2p)!(m-2p)!}\left\|\lambda D_y^{m-2p}v\right\|_{L^2}\right)\\ &\qquad+C_0\lambda^p\sum_{j=2}^{2p-1}\frac{m!}{(m-j)!}\left(\left\|Z_\lambda D_y^{m-j}v\right\|_{L^2}+\left\|\lambda D_y^{m-j}v\right\|_{L^2}+\left\|\lambda D_y^{m-j}v\right\|_{L^2}\right)\\ &\qquad+C_0\lambda^p\sum_{j=3}^{p}\frac{(m-1)!}{(m-j)!}\left(\left\|Z_\lambda D_y^{m-j+1}v\right\|_{L^2}+\left\|\lambda D_y^{m-j+2}v\right\|_{L^2}\right).\end{aligned}$$

The proof of the proposition rests on the following series of lemmas.

Lemma 3.2 *Let $p,q\ge 1$ and let $L_{p,q;\lambda}$ be the operator given in (2.3). Then there exists a constant C_0, depending only on p,q and the constant C given in (2.1), such that for every integer $m\ge 1$ and any given $v\in H^\infty\bigcap\mathcal{H}^\infty_{Z_\lambda}$, we have*

$$\begin{aligned}\left\|\lambda D_y^m v\right\|_{L^2}+\left\|Z_\lambda D_y^m v\right\|_{L^2}\le\ & C_0\left\|D_y^m L_{p,q;\lambda}v\right\|_{L^2}+C_0 m\left\|\lambda^{p+1}y^{p-1}D_y^{m-1}v\right\|_{L^2}\\ &+C_0\lambda^p\tfrac{m!}{(2p)!(m-2p)!}\left\|\lambda D_y^{m-2p}v\right\|_{L^2}\\ &+C_0\textstyle\sum_{j=2}^{2p-1}\frac{m!}{j!(m-j)!}\left\|\lambda^{p+1}y^{\delta_j}D_y^{m-j}v\right\|_{L^2}\\ &+C_0\textstyle\sum_{j=3}^{p}\frac{(m-1)!}{j!(m-j)!}\left\|\lambda^{p+1}y^{\rho_j}D_y^{m-j+1}v\right\|_{L^2},\end{aligned}\tag{3.3}$$

where $\delta_j, \rho_j \in \{1, 2, \ldots, p-1\}$. *Similarly we have*

$$\begin{aligned}\left\|\lambda D_x^m v\right\|_{L^2} + \left\|Z_\lambda D_x^m v\right\|_{L^2} \le C_0\left\|D_x^m L_{p,q;\lambda} v\right\|_{L^2} + C_0 m\left\|\lambda^{q+1} y^{q-1} D_x^{m-1} v\right\|_{L^2}\\ + C_0\lambda^q \frac{m!}{(2q)!(m-2q)!}\left\|\lambda D_x^{m-2q} v\right\|_{L^2}\\ + C_0\sum_{j=2}^{2q-1}\frac{m!}{j!(m-j)!}\left\|\lambda^{q+1} y^{\delta_j} D_x^{m-j} v\right\|_{L^2} \quad (3.4)\\ + C_0\sum_{j=3}^{q}\frac{(m-1)!}{j!(m-j)!}\left\|\lambda^{q+1} y^{\rho_j} D_x^{m-j+1} v\right\|_{L^2}.\end{aligned}$$

where $\delta_j, \rho_j \in \{1, 2, \ldots, q-1\}$.

Proof We need only prove (3.3) since (3.4) can be handled in a similar way. To simplify the notation, we use C_p to denote different suitable constants, which depend only on p and the constant C in (2.1). Leibniz formula gives

$$L_{p,q;\lambda} D_y^m v = D_y^m L_{p,q;\lambda} v - \sum_{j=1}^{2p}\binom{m}{j}\left(D_y^j\left(\lambda^{2(p+1)} y^{2p}/4\right) - D_y^j\left(\lambda^{p+1} y^p\right) D_x\right) D_y^{m-j} v.$$

On the other hand, a direct computation shows that

$$\begin{aligned}&D_y^j\left(\lambda^{2(p+1)} y^{2p}/4\right) - D_y^n\left(\lambda^{p+1} y^p\right) D_x\\ &= \begin{cases} i\lambda^{p+1} p y^{p-1}\left(D_x - \lambda^{p+1} y^p/2\right), & j = 1,\\ a_{p,j}\lambda^{p+1} y^{p-j}\left(D_x - \lambda^{p+1} y^p/2\right) + b_{p,j}\lambda^{2(p+1)} y^{2p-j}, & 2 \le j \le p,\\ c_{p,j}\lambda^{2(p+1)} y^{2p-j}, & p < j \le 2p,\end{cases}\end{aligned}$$

where $a_{p,j}, b_{p,j}, c_{p,j}$ are constants depending only on p and j such that

$$|a_{p,j}| + |b_{p,j}| + |c_{p,j}| \le C_p. \quad (3.5)$$

Then

$$\begin{aligned}L_{p,q;\lambda} D_y^m v = D_y^m L_{p,q;\lambda} v - imp\lambda^{p+1} y^{p-1}\left(D_x - \lambda^{p+1} y^p/2\right) D_y^{m-1} v\\ - \sum_{j=2}^{p}\binom{m}{j} a_{p,j}\lambda^{p+1} y^{p-j}\left(D_x - \lambda^{p+1} y^p/2\right) D_y^{m-j} v\\ - \sum_{j=2}^{p}\binom{m}{j}\lambda^{2(p+1)} b_{p,j} y^{2p-j} D_y^{m-j} v\\ - \sum_{j=p+1}^{2p}\binom{m}{j}\lambda^{2(p+1)} c_{p,j} y^{2p-j} D_y^{m-j} v.\end{aligned}$$

Taking the L^2-inner product with $D_y^m v$ on both sides and using (2.1) gives

$$\left\|\lambda D_y^m v\right\|^2+\left\|(D_x-\lambda^{p+1}y^p/2)D_y^m v\right\|^2+\left\|(D_y+\lambda^{q+1}x^q/2)D_y^m v\right\|^2\leq C_p\sum_{1\leq j\leq 5} I_j, \tag{3.6}$$

where

$$\begin{aligned}
I_1 &= \left(D_y^m L_{p,q;\lambda}v,\ D_y^m v\right)_{L^2},\\
I_2 &= -i\,m\,p\left(\lambda^{p+1}y^{p-1}\left(D_x-\lambda^{p+1}y^p/2\right)D_y^{m-1}v,\ D_y^m v\right)_{L^2},\\
I_3 &= -\sum_{j=2}^{p}\binom{m}{j}a_{p,j}\left(\lambda^{p+1}y^{p-j}\left(D_x-\lambda^{p+1}y^p/2\right)D_y^{m-j}v,\ D_y^m v\right)_{L^2},\\
I_4 &= -\sum_{j=2}^{p}\binom{m}{j}b_{p,j}\left(\lambda^{2(p+1)}y^{2p-j}D_y^{m-j}v,\ D_y^m v\right)_{L^2},\\
I_5 &= -\sum_{j=p+1}^{2p}\binom{m}{j}c_{p,j}\left(\lambda^{2(p+1)}y^{2p-j}D_y^{m-j}v,\ D_y^m v\right)_{L^2}.
\end{aligned}$$

In view of (3.5), a direct computation gives that for any given $\varepsilon>0$ there exists a constant $C_{\varepsilon,p}$, depending only on ε and p, such that

$$I_1\leq\left\|D_y^m v\right\|_{L^2}\left\|D_y^m L_{p,q;\lambda}v\right\|_{L^2}\leq\varepsilon\left\|\lambda D_y^m v\right\|_{L^2}^2+\lambda^{-1}C_{\varepsilon,p}\left\|D_y^m L_{p,q;\lambda}v\right\|_{L^2}^2,$$

and

$$\begin{aligned}
I_2+I_3\leq\ &\varepsilon\left\|\left(D_x-\lambda^{p+1}y^p/2\right)D_y^m v\right\|^2+C_{\varepsilon,p}m^2\left\|\lambda^{p+1}y^{p-1}D_y^{m-1}v\right\|_{L^2}^2\\
&+C_{\varepsilon,p}\sum_{j=2}^{p}\binom{m}{j}^2\left\|\lambda^{p+1}y^{p-j}D_y^{m-j}v\right\|_{L^2}^2.
\end{aligned}$$

For the term I_4 we have

$$\begin{aligned}
I_4\leq\ &C_p m^2\left\|\lambda^{p+1}y^{p-1}D_y^{m-1}v\right\|_{L^2}^2+C_p\sum_{j=2}^{p}\left(\frac{(m-1)!}{j!(m-j)!}\right)^2\left\|\lambda^{p+1}y^{p-j}D_y^{m-j}v\right\|_{L^2}^2\\
&+C_p\sum_{j=3}^{p}\left(\frac{(m-1)!}{j!(m-j)!}\right)^2\left\|\lambda^{p+1}y^{p-j+1}D_y^{m-j+1}v\right\|_{L^2}^2.
\end{aligned}$$

To see this, we integrate by parts to get

$$\begin{aligned}
I_4=&-\sum_{j=2}^{p}\binom{m}{j}b_{p,j}\left(\left(D_y\left(\lambda^{2(p+1)}y^{2p-j}\right)\right)D_y^{m-j}v,\ D_y^{m-1}v\right)_{L^2}\\
&-\sum_{j=2}^{p}\binom{m}{j}b_{p,j}\left(\lambda^{2(p+1)}y^{2p-j}D_y^{m-j+1}v,\ D_y^{m-1}v\right)_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{j=2}^{p} \binom{m}{j} b_{p,j} (2p-j) \Big(\lambda^{2(p+1)} y^{2p-j-1} D_y^{m-j} v,\ D_y^{m-1} v\Big)_{L^2} \\
&\quad - b_{p,2} \frac{m(m-1)}{2} \big\|\lambda^{p+1} y^{p-1} D_y^{m-1} v\big\|_{L^2}^2 \\
&\quad - \sum_{j=3}^{p} \binom{m}{j} b_{p,j} \Big(\lambda^{2(p+1)} y^{2p-j} D_y^{m-j+1} v,\ D_y^{m-1} v\Big)_{L^2},
\end{aligned}
$$

which implies the desired upper bound for I_4. Arguing similarly, we get the estimate for I_5:

$$
\begin{aligned}
I_5 \le\ & C_p m^2 \big\|\lambda^{p+1} D_y^{m-1} v\big\|_{L^2}^2 + C_p \sum_{j=p+1}^{2p} \left(\frac{(m-1)!}{j!(m-j)!}\right)^2 \big\|\lambda^{p+1} y^{2p-j} D_y^{m-j} v\big\|_{L^2}^2 \\
& + C_p \sum_{j=p+1}^{2p} \left(\frac{(m-1)!}{j!(m-j)!}\right)^2 \big\|\lambda^{p+1} y^{p-j+1} D_y^{m-j+1} v\big\|_{L^2}^2.
\end{aligned}
$$

Combining the estimates on $I_1, I_2, \ldots, I_5$ and the inequality (3.6), the conclusion (3.3) follows immediately if we let ε above be sufficiently small. The proof is thus complete. □

Lemma 3.3 *Let $q = 1$, $p \ge 1$. For any given $\varepsilon > 0$, there exists a constant C_ε, which depends only on ε and p, such that for every $m \ge 1$ and any given $v \in H^\infty$ we have*

$$
\begin{aligned}
& m \big\|\lambda^{p+1} y^{p-1} D_y^{m-1} v\big\|_{L^2} \\
& \quad \le \varepsilon\lambda \big\|D_x^m v\big\| + \varepsilon\lambda \big\|D_y^m v\big\| + C_\varepsilon m \lambda^p \big\|\lambda D_y^{m-1} v\big\| + C_\varepsilon \lambda m^{(p+1)/2} \big\|Z_\lambda D_y^{m-1} v\big\|.
\end{aligned}
\tag{3.7}
$$

Proof For any given $\varepsilon > 0$, we compute

$$
\begin{aligned}
& m^2 \big\|\lambda^{p+1} y^{p-1} D_y^{m-1} v\big\|_{L^2}^2 \\
& \quad = m^2 \left(\int_{|y|\le 2} + \int_{|y|\ge \varepsilon^{-1}\lambda^{-1} m} + \int_{2<|y|<\varepsilon^{-1}\lambda^{-1} m} \right) \lambda^{2(p+1)} y^{2(p-1)} \big|D_y^{m-1} v\big|^2 dx dy \\
& \quad \le m^2 2^{2(p-1)} \lambda^{2(p+1)} \int_{|y|\le 2} \big|D_y^{m-1} v\big|^2 dx dy + \varepsilon^2 \lambda^2 \int_{|y|\ge \varepsilon^{-1}\lambda^{-1} m} \lambda^{2(p+1)} y^{2p} \big|D_y^{m-1} v\big|^2 dx dy \\
& \quad\quad + m^2 \varepsilon^{-(p-1)} \lambda^2 m^{p-1} \int_{2<|y|<\varepsilon^{-1}\lambda^{-1} m} \lambda^{p+1} |y|^{p-1} \big|D_y^{m-1} v\big|^2 dx dy \\
& \quad \le 4^{p-1} m^2 \lambda^{2(p+1)} \big\|D_y^{m-1} v\big\|^2 + \varepsilon^2 \lambda^2 \big\|\lambda^{p+1} y^p D_y^{m-1} v\big\|^2
\end{aligned}
$$

$$+C_\varepsilon \lambda^2 m^{p+1} \int_{2<|y|<\varepsilon^{-1}\lambda^{-1}m} \lambda^{p+1}|y|^{p-1}\left|D_y^{m-1}v\right|^2 dxdy$$
$$\leq 4^{p-1}m^2\lambda^{2p}\left\|\lambda D_y^{m-1}v\right\|^2 + \varepsilon^2\lambda^2\left\|\lambda^{p+1}y^p D_y^{m-1}v\right\|^2$$
$$+2C_\varepsilon\lambda^2 m^{p+1}\left\||\lambda^{q+1}x^{q-1}+\lambda^{p+1}y^{p-1}|^{1/2}D_y^{m-1}v\right\|^2,$$

where the last inequality holds because when $2 < |y|, \lambda \geq 1, p \geq 1$ and $q = 1$ $(x \neq 0)$ one has

$$\frac{1}{2}\lambda^{p+1}|y|^{p-1} \leq \left|\lambda^{q+1}x^{q-1}+\lambda^{p+1}y^{p-1}\right| = \left|\lambda^2+\lambda^{p+1}y^{p-1}\right|. \tag{3.8}$$

Furthermore,

$$\left\|\lambda^{p+1}y^p D_y^{m-1}v\right\|^2 \leq 2\left\|\left(D_x-\lambda^{p+1}y^p\right)D_y^{m-1}v\right\|^2+2\left\|D_xD_y^{m-1}v\right\|^2$$
$$\leq 2\left\|\left(D_x-\lambda^{p+1}y^p\right)D_y^{m-1}v\right\|^2+2\left\|D_y^m v\right\|^2+2\left\|D_x^m v\right\|^2, \tag{3.9}$$

and

$$\left\||\lambda^2+\lambda^{p+1}y^{p-1}|^{1/2}D_y^{m-1}v\right\|^2 \leq C\left\|Z_\lambda D_y^{m-1}v\right\|^2$$

by (2.1). Combining these inequalities yields (3.7). □

Lemma 3.4 *Let $p, q \geq 2$. Then for any given $\varepsilon > 0$, there exists a constant C_ε, which depends only on ε, p and the constant C in (2.1), such that for every $m \geq 1$ and all $v \in H^\infty$ we have*

$$m\left\|\lambda^{p+1}y^{p-1}D_y^{m-1}v\right\|_{L^2} \leq \varepsilon\left(\left\|D_y^m v\right\|_{L^2}+\left\|D_x^m v\right\|_{L^2}\right)$$
$$+C_\varepsilon\lambda^p m^{(2p+2)/3}\left\|Z_\lambda D_y^{m-1}v\right\|_{L^2}. \tag{3.10}$$

In particular, if both p and q are odd then

$$m\left\|\lambda^{p+1}y^{p-1}D_y^{m-1}v\right\|_{L^2} \leq \varepsilon\left(\left\|D_y^m v\right\|_{L^2}+\left\|D_x^m v\right\|_{L^2}\right)$$
$$+C_\varepsilon\lambda^p m^{(p+1)/2}\left\|Z_\lambda D_y^{m-1}v\right\|_{L^2}. \tag{3.11}$$

Proof The proof of (3.11) is quite similar to that of the previous lemma, since the estimate (3.8) still holds if both p and q are odd. As for (3.10), the condition $p \geq 2$ yields that, for any given $\varepsilon > 0$,

$$m^2\left\|\lambda^{p+1}y^{p-1}D_y^{m-1}v\right\|_{L^2}^2$$
$$= m^2\left(\int_{|y|<m/\varepsilon}+\int_{|y|\geq m/\varepsilon}\right)\lambda^{2(p+1)}|y|^{2(p-1)}|D_y^{m-1}v|^2 dxdy$$
$$\leq \varepsilon^2\left\|\lambda^{p+1}y^p D_y^{m-1}v\right\|_{L^2}^2+\varepsilon^{-(4p-2)/3}m^{(4p+4)/3}\lambda^{4(p+1)/3}\left\|\lambda^{(p+1)/3}y^{(p-2)/3}D_y^{m-1}v\right\|_{L^2}^2.$$

One then bounds the right-hand side terms by using (3.9) and (2.2), thus obtaining (3.10). □

Lemma 3.5 *For all k, ℓ with $\ell \leq p-1$ and $2 \leq k \leq m$, we have*

$$\begin{aligned}&\left\|\lambda^{p+1} y^{\ell} D_y^{m-k} v\right\|_{L^2}\\ &\quad \leq C_p\left(\left\|Z_\lambda D_y^{m-k} v\right\|_{L^2}+\left\|D_x^{m-k+1} v\right\|_{L^2}+\left\|D_y^{m-k+1} v\right\|_{L^2}+\lambda^p\left\|\lambda D_y^{m-k} v\right\|_{L^2}\right).\end{aligned} \tag{3.12}$$

Proof This is just a consequence of (3.9), since, by splitting the integration on the regions $\{|y| \leq 1\}$ and $\{|y| \geq 1\}$ and using the fact that $\ell \leq p-1$, we have

$$\left\|\lambda^{p+1} y^{\ell} D_y^{m-k} v\right\|_{L^2}^2 \leq\left\|\lambda^{p+1} y^p D_y^{m-k} v\right\|_{L^2}^2+\lambda^{2(p+1)}\left\|D_y^{m-k} v\right\|_{L^2}^2.$$

□

Proof of Proposition 3.1 The first conclusion for $p, q \geq 2$ in Proposition 3.1 follows from (3.3, 3.4, 3.10) and (3.12), while the second conclusion is just a consequence of (3.3, 3.4, 3.10) and (3.7). The proof is complete. □

4 Global hypoellipticity

4.1 Global hypoellipticity in Sobolev spaces

In this section we prove Theorem 1.2, that is, the global H^m-hypoellipticity in $\mathbb{R}^{2n}$. As already mentioned, for simplicity we will give a proof when $n = 1$. In view of Proposition 2.1 it suffices to prove the following proposition.

Proposition 4.1 *Let $L_{p,q;\lambda}$ be the operator given in (2.3). Then for any given non-negative integer m, we have*

$$u \in L^2,\ L_{p,q;\lambda} u \in H^m \Longrightarrow u \in H^m \bigcap \mathcal{H}_{Z_\lambda}^m .$$

Proof **Step 1.** Since $u \in L^2$, there exits a sequence $\{u_k\}_{k \geq 1} \subset C_0^{\infty}(\mathbb{R}^2)$ such that $u_k \xrightarrow{L^2} u$ as $k \rightarrow +\infty$. Hence

$$\sup_{k \geq 1}\left\|u_k\right\|_{L^2}<+\infty, \tag{4.1}$$

and, for every $\varphi \in C_0^{\infty}(\mathbb{R}^2)$,

$$\begin{aligned}\left(D^\alpha L_{p,q;\lambda} u_k,\ \varphi\right)_{L^2} &=\left(u_k,\ L_{p,q;\lambda} D^\alpha \varphi\right)_{L^2} \longrightarrow\left(u,\ L_{p,q;\lambda} D^\alpha \varphi\right)_{L^2}\\ &=\left(D^\alpha L_{p,q;\lambda} u,\ \varphi\right)_{L^2}, \text{ as } k \rightarrow \infty,\end{aligned}$$

where α is an arbitrary multi-index such that $|\alpha| \leq m$. This shows that for any given $|\alpha| \leq m$ the sequence $\{D^\alpha L_{p,q;\lambda} u_k\}_{k\geq 1}$ converges weakly to $D^\alpha L_{p,q;\lambda} u$ in L^2. Therefore

$$\sum_{|\alpha|\leq m} \sup_{k\geq 1} \left\| D^\alpha L_{p,q;\lambda} u_k \right\|_{L^2} < +\infty. \tag{4.2}$$

Step 2. In this step we use induction to show that

$$\forall\, |\alpha| \leq m, \quad \sup_{k\geq 1} \left\| \lambda D^\alpha u_k \right\|_{L^2} + \sup_{k\geq 1} \left\| Z_\lambda D^\alpha u_k \right\|_{L^2} < +\infty. \tag{4.3}$$

By virtue of (4.1) and (4.2), the above estimate holds for $\alpha = 0$, since, in view of the relation

$$\left\| Z_\lambda u_k \right\|_{L^2}^2 = \left(L_{p,q;\lambda} u_k,\ u_k \right)_{L^2},$$

one has

$$\sup_{k\geq 1} \left\| Z_\lambda u_k \right\|_{L^2} \leq \sup_{n\geq 1} \left\| L_{p,q;\lambda} u_k \right\|_{L^2}^{1/2} \left\| u_k \right\|_{L^2}^{1/2}.$$

Moreover, from the validity of (4.3) when $\alpha = 0$, applying Proposition 3.1 with $m = 1$ gives

$$\sup_{k\geq 1} \left\| \lambda D_x u_k \right\|_{L^2} + \sup_{k\geq 1} \left\| \lambda D_y u_k \right\|_{L^2} + \sup_{k\geq 1} \left\| Z_\lambda D_x u_k \right\|_{L^2} + \sup_{k\geq 1} \left\| Z_\lambda D_y u_k \right\|_{L^2} < +\infty.$$

Then using again Proposition 3.1 repeatedly, a standard iteration yields

$$\forall\, r \leq m, \quad \sup_{k\geq 1} \left\| \lambda D_x^r u_k \right\|_{L^2} + \sup_{k\geq 1} \left\| \lambda D_y^r u_k \right\|_{L^2} + \sup_{k\geq 1} \left\| Z_\lambda D_x^r u_k \right\|_{L^2} + \sup_{k\geq 1} \left\| Z_\lambda D_y^r u_k \right\|_{L^2} < +\infty,$$

or, equivalently,

$$\forall\, |\alpha| \leq m,\ \forall\, r \leq m, \quad \sup_{k\geq 1} \left\| \lambda D^\alpha u_k \right\|_{L^2} + \sup_{k\geq 1} \left\| Z_\lambda D_x^r u_k \right\|_{L^2} + \sup_{k\geq 1} \left\| Z_\lambda D_y^r u_k \right\|_{L^2} < +\infty. \tag{4.4}$$

We now complete the proof of (4.3), and assume

$$\forall\, |\beta| \leq m-1, \quad \sup_{k\geq 1} \left\| Z_\lambda D^\beta u_k \right\|_{L^2} < +\infty. \tag{4.5}$$

We have to prove the validity of (4.5) when $|\beta| = m$. To do so, let us choose an arbitrary multi-index β such that $|\beta| = m$. Then

$$
\begin{aligned}
\left\|Z_{j,\lambda}D^{\beta}u_k\right\|_{L^2} &\leq \left\|D^{\beta}Z_{j,\lambda}u_k\right\|_{L^2} + \left\|\left[D^{\beta},\ Z_{j,\lambda}\right]u_k\right\|_{L^2}\\
&\leq \left\|D_x^m Z_{j,\lambda}u_k\right\|_{L^2} + \left\|D_y^m Z_{j,\lambda}u_k\right\|_{L^2} + \left\|\left[D^{\beta},\ Z_{j,\lambda}\right]u_k\right\|_{L^2}\\
&\leq \left\|Z_{j,\lambda}D_x^m u_k\right\|_{L^2} + \left\|Z_{j,\lambda}D_y^m u_k\right\|_{L^2}\\
&\quad + \left\|\left[D_x^m,\ Z_{j,\lambda}\right]u_k\right\|_{L^2} + \left\|\left[D_y^m,\ Z_{j,\lambda}\right]u_k\right\|_{L^2} + \left\|\left[D^{\beta},\ Z_{j,\lambda}\right]u_k\right\|_{L^2}.
\end{aligned}
$$

Observe that $\left[D_x^m,\ Z_{j,\lambda}\right]$, $\left[D_y^m,\ Z_{j,\lambda}\right]$ and $\left[D^{\beta},\ Z_{j,\lambda}\right]$ can be written as linear combinations of the terms

$$
\lambda^{q+1}x^{q-k}D^{\beta'},\ \lambda^{p+1}y^{p-k}D^{\beta''},\quad 1 \leq k, \ell \leq p,\ 0 \leq |\beta'|, |\beta''| \leq m-1.
$$

Then by (3.12) we have

$$
\begin{aligned}
&\left\|\left[D_x^m,\ Z_{j,\lambda}\right]u_k\right\|_{L^2} + \left\|\left[D_y^m,\ Z_{j,\lambda}\right]u_k\right\|_{L^2} + \left\|\left[D^{\beta},\ Z_{j,\lambda}\right]u_k\right\|_{L^2}\\
&\leq \sum_{|\gamma|\leq m-1} C_{\gamma,p,q}\left\|Z_{j,\lambda}D^{\gamma}u_k\right\|_{L^2} + \sum_{|\gamma|\leq m} C_{\gamma,p,q}\left\|D^{\gamma}u_k\right\|_{L^2},
\end{aligned}
$$

with $C_{\gamma,p,q}$ constants depending only on γ, p and q. Combining these inequalities yields, for any given $|\beta| = m$,

$$
\begin{aligned}
\left\|Z_{j,\lambda}D^{\beta}u_k\right\|_{L^2} \leq{}& \left\|Z_{j,\lambda}D_x^m u_k\right\|_{L^2} + \left\|Z_{j,\lambda}D_y^m u_k\right\|_{L^2} + \sum_{|\gamma|\leq m-1} C_{\gamma,p,q}\left\|Z_{j,\lambda}D^{\gamma}u_k\right\|_{L^2}\\
&+ \sum_{|\gamma|\leq m} C_{\gamma,p,q}\left\|D^{\gamma}u_k\right\|_{L^2}.
\end{aligned}
$$

Thus, in view of (4.4) and the induction assumption (4.5), we get

$$
\forall\ |\beta| \leq m,\quad \sup_{k\geq 1}\left\|Z_{\lambda}D^{\beta}u_k\right\|_{L^2} < +\infty,
$$

and (4.3) follows.

Step 3. We now prove that $u \in H^m \bigcap \mathcal{H}^m_{Z_\lambda}$. Let α be an arbitrary index such that $|\alpha| \leq m$. The uniform boundedness of the sequence $\{D^{\alpha}u_k\}_{k\geq 1}$, granted by (4.3), allows us to find a weakly convergent subsequence of $\{D^{\alpha}u_k\}_{k\geq 1}$, still denoted by $\{D^{\alpha}u_k\}_{k\geq 1}$. Hence, there exists an element $G_{\alpha} \in L^2$ such that

$$
\forall\ \varphi \in C_0^{\infty}(\mathbb{R}^2),\quad \lim_{k\to+\infty}\left(D^{\alpha}u_k,\ \varphi\right)_{L^2} = (G_{\alpha},\ \varphi)_{L^2}\,.
$$

On the other hand, since $\{u_k\}_{k\geq 1}$ converges strongly to u in L^2, we then have

$$
\forall\ \varphi \in C_0^{\infty}(\mathbb{R}^2),\quad \lim_{k\to+\infty}\left(D^{\alpha}u_k,\ \varphi\right)_{L^2} = \lim_{k\to+\infty}\left(u_k,\ D^{\alpha}\varphi\right)_{L^2} = \left(u,\ D^{\alpha}\varphi\right)_{L^2}\,.
$$

This implies $D^\alpha u = G_\alpha \in L^2$, and thus $u \in H^m$.

In view of (4.3), we can argue as above to conclude that for each $j = 1, 2$, and for any given α with $|\alpha| \le m$, $Z_{j,\lambda} D^\alpha u \in L^2$ and $\{Z_{j,\lambda} D^\alpha u_k\}_{\ge 1}$ converges weakly to $Z_{j,\lambda} D^\alpha u$ in L^2. The proof is thus complete. □

4.2 Global Gevrey hypoellipticity

We now prove Theorem 1.3. By the H^m-hypoellipticity of $L_{p,q;\lambda}$, we have $u \in H^\infty \bigcap \mathcal{H}^\infty_{Z_\lambda}$ if $L_{p,q;\lambda} u = f$ with $f \in H^\infty$. Starting from the H^∞-smooth solution u we have the following two properties which give the global Gevrey hypoellipticity.

Proposition 4.2 *Let $p, q \ge 2$ and let $L_{p,q;\lambda}$ be the operator given in (2.3). Denote*

$$\sigma = \begin{cases} \max\{(p+1)/2, (q+1)/2\}, & \text{if both } p \text{ and } q \text{ are odd;} \\ \max\{(2p+2)/3, (2q+2)/3\}, & \text{otherwise.} \end{cases}$$

Suppose $u \in H^\infty \bigcap \mathcal{H}^\infty_{Z_\lambda}$ such that $L_{p,q;\lambda} u = f$ with $f \in C^\infty$ satisfying the condition

$$\forall\, k \in \mathbb{Z}_+, \quad \left\| D_x^k f \right\|_{L^2} + \left\| D_y^k f \right\|_{L^2} \le M_1^{k+1} (k!)^\sigma ,$$

for some constant M_1. Then there exists a constant C_0, depending only on p, q, M_1 and the constant C given in (2.1), such that for given $m \ge 1$, if

$$\forall\, k \le m-1, \quad \left\| \lambda D_x^k u \right\|_{L^2} + \left\| \lambda D_y^k u \right\|_{L^2} + \left\| Z_\lambda D_x^k u \right\|_{L^2} + \left\| Z_\lambda D_y^k u \right\|_{L^2} \le \lambda^{(p+q)k} M^{k+1} (k!)^\sigma$$

for some constant $M \ge M_1$, then

$$\left\| \lambda D_x^m u \right\|_{L^2} + \left\| \lambda D_y^m u \right\|_{L^2} + \left\| Z_\lambda D_x^m u \right\|_{L^2} + \left\| Z_\lambda D_y^m u \right\|_{L^2} \le C_0 \lambda^{(p+q)m} M^{m+1} (m!)^\sigma .$$

As a result, if we choose $M \ge M_1 + C_0 + \left\| u \right\|_{L^2}$, then we have by induction that

$$\forall\, k \ge 0, \ \left\| \lambda D_x^k u \right\|_{L^2} + \left\| \lambda D_y^k u \right\|_{L^2} + \left\| Z_\lambda D_x^k u \right\|_{L^2} + \left\| Z_\lambda D_y^k u \right\|_{L^2} \le \lambda^{(p+q)k} M^{k+1} (k!)^\sigma ,$$

and thus $L_{p,q;\lambda}$ is globally $G^{\sigma,\sigma}$-hypoelliptic in $\mathbb{R}^2$.

Proof This is just an immediate consequence of Proposition 3.1. □

Similarly we have

Proposition 4.3 *Let $q = 1$ and $p \ge 1$, and let $L_{p,q;\lambda}$ be the operator given in (2.3). Suppose $u \in H^\infty \bigcap \mathcal{H}^\infty_{Z_\lambda}$ such that $L_{p,q;\lambda} u = f$ with $f \in C^\infty$ satisfying the following condition:*

$$\forall\, k, \ell \in \mathbb{Z}_+, \quad \left\| D_x^k D_y^\ell f \right\|_{L^2} \le M_1^{k+\ell+1} k! \, (\ell!)^{(p+1)/2} ,$$

for some constant M_1. Then there exists a constant C_0, depending only on p, q, M_1 and the constant C given in (2.1), such that for given $m \geq 2p + 2q$, if

$$\forall\, k \leq m-1, \quad \left\|D_x^k u\right\|_{L^2} \leq \lambda^k M^{k+1} k!, \quad \left\|D_y^k u\right\|_{L^2} \leq \lambda^{pk} M^{k+1} (k!)^{(p+1)/2},$$

for some constant $M \geq M_1$, then

$$\left\|\lambda D_x^m u\right\|_{L^2} + \left\|Z_\lambda D_x^m u\right\|_{L^2} \leq C_0 \lambda^m M^m m!$$

and

$$\left\|\lambda D_y^m u\right\|_{L^2} + \left\|Z_\lambda D_y^m u\right\|_{L^2} \leq C_0 \lambda^{pm} M^m (m!)^{(p+1)/2}.$$

As a result, $L_{p,q;\lambda}$ is globally $G^{1,(p+1)/2}$-hypoelliptic in $\mathbb{R}^2$.

5 The case when $p = q = 1$

As seen in Theorem 1.3, in this case we have that the twisted Laplacian L is globally analytic hypoelliptic. We give in this section a direct proof of it, which uses the Green's function of L.

Theorem 5.1 *The twisted Laplacian L is globally analytic hypoelliptic in $\mathbb{R}^{2n}$.*

Again, we shall prove this for $n = 1$. Recall that the Green's function of L is given by

$$G(z, w) = \frac{1}{4\pi} e^{i\sigma(z,w)/4} \int_0^\infty e^{-\frac{1}{4}|z-w|^2 \cosh t}\, dt,$$

where $z = (x_1, y_1)$, $w = (x_2, y_2)$, and $\sigma(z, w) = x_2 y_1 - x_1 y_2$ is the canonical symplectic 2-form in $\mathbb{R}^2$.

Proof Let $Lu = f$. Then

$$u(z) = \int_{\mathbb{R}^2} G(z, w) f(w)\, dw = \int_{\mathbb{R}^2} g(w) e^{i\sigma(z,w)/4} f(z-w)\, dw,$$

where

$$g(w) = \frac{1}{4\pi} \int_0^\infty e^{-\frac{1}{4}|z|^2 \cosh t}\, dt.$$

For any given multi-index α we have

$$D_z^\alpha u(z) = \int_{\mathbb{R}^2} g(w) D_z^\alpha \left(e^{i\sigma(z,w)/4} f(z-w)\right) dw$$
$$= \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \int_{\mathbb{R}^2} g(w) \left(D_z^{\alpha-\beta} e^{i\sigma(z,w)/4}\right) D_z^\beta f(z-w)\, dw.$$

Hence

$$\left|D_z^\alpha u(z)\right| \le \sum_{\beta\le\alpha} \binom{\alpha}{\beta} H_{\alpha,\beta} * \left|D_z^\beta f\right|(z),$$

with $H_{\alpha,\beta}(z) = g(z)\left|z\right|^{|\alpha|-|\beta|}$, and thus by Young's inequality for convolutions,

$$\left\|D_z^\alpha u\right\|_{L^2} \le \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \left\|H_{\alpha,\beta}\right\|_{L^1} \left\|D_y^\beta f\right\|_{L^2}. \tag{5.1}$$

In view of (1.2),

$$\left\|D_z^\beta f\right\|_{L^2} \le M_1^{|\beta|+1}\beta!,$$

for some constant $M_1 > 1$. On the other hand, for $|z| \ge 1$ we have

$$\int_1^\infty |z|^k e^{-\frac{1}{4}|z|^2\cosh t}\, dt \le \int_1^\infty |z|^k e^{-|z|^2t^2/8}\, dt = |z|^{k-1}\int_{|z|}^\infty e^{-s^2/8} ds \le 8|z|^{k-1}e^{-|z|/8},$$

and

$$\int_0^1 |z|^k e^{-\frac{1}{4}|z|^2\cosh t}\, dt \le \int_0^1 |z|^k e^{-|z|^2/4}\, dt = |z|^k e^{-|z|^2/4} \le 8|z|^k e^{-|z|/8}.$$

This implies

$$\forall\, |z| \ge 1,\ \forall\, k\in\mathbb{N},\ \ g(z)|z|^k = \int_1^\infty |z|^k e^{-\frac{1}{4}|z|^2\cosh t}\, dt \le 16|z|^k e^{-|z|/8} \le |z|^{-3}\tilde{M}^{k+1}k!,$$

for some constant $\tilde{M} > 1$. So

$$\left\|H_{\alpha,\beta}\right\|_{L^1} = \int_{|z|\le 1} g(z)|z|^{|\alpha|-|\beta|}\, dz + \int_{|z|\ge 1} g(z)|z|^{|\alpha|-|\beta|}\, dz$$
$$\le C + \tilde{M}^{|\alpha|-|\beta|+1}(|\alpha|-|\beta|)! \int_{|z|\ge 1} |z|^{-3}\, dz$$

$$\le M_2^{|\alpha|-|\beta|+1}(|\alpha|-|\beta|)!$$
$$\le (2M_1+M_2)^{|\alpha|-|\beta|+1}(|\alpha|-|\beta|)!$$

where M_1 is the constant given in (1.2). As for any given multi-index γ we have $|\gamma|! \le n^{|\gamma|}\gamma!$, we therefore get from (5.1) that

$$\left\| D_z^\alpha u \right\|_{L^2} \le (2M_1+M_2)^{|\alpha|+1} \sum_{\beta\le\alpha} \binom{\alpha}{\beta} 2^{|\alpha|-|\beta|} \left(\frac{M_1}{2M_1+M_2}\right)^{|\beta|} (\alpha-\beta)!,$$

and finally that

$$\left\| D_z^\alpha u \right\|_{L^2} \le (4M_1+2M_2)^{|\alpha|+1}\,\alpha! \sum_{\beta\le\alpha} 4^{-|\beta|} \le M^{|\alpha|+1}\alpha!,$$

for some constant M depending only on M_1, M_2. This conlcudes the proof. □

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