Global hypoelliptic estimates for fractional order kinetic equation

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In this paper we study a class of fractional order kinetic equation, which is a linear model of spatially inhomogeneous Boltzmann equation without angular cutoff. Using the multiplier method introduced by F. Hérau and K. Pravda-Starov (J. Math. Pures et Appl., 2011), we establish the optimal global hypoelliptic estimates with weights for the linear model operator.

1 Introduction and main results

Inspired by the work of Hérau and Pravda-Starov [22] on the global hypoellipticity of Landau-type operator, we study in this paper the hypoellipticity of a class of fractional order kinetic equation, which is a linear model of spatially inhomogeneous Boltzmann equation without angular cutoff and takes the following form:

\[ P = \partial_t + v \cdot \partial_x + a(v) \left( -\tilde{\triangle} v \right)^s + b(v), \quad 0 < s \leq 1, \]

where the coefficients \( a, b \) are smooth real-valued functions of the velocity variable \( v \) with the properties subsequently listed below. There exist a number \( \gamma \in \mathbb{R} \) and a constant \( C \geq 1 \) such that for all \( v \in \mathbb{R}^n \) we have

\[ C^{-1} \langle v \rangle^\gamma \leq a(v) \leq C \langle v \rangle^{2s+\gamma}, \quad C^{-1} \langle v \rangle^{2s+\gamma} \leq b(v) \leq C \langle v \rangle^{2s+\gamma}, \]

and

\[ \forall \ |\alpha| \geq 0, \exists C_\alpha > 0, \quad |\partial_\alpha^a a(v)| + |\partial_\alpha^b b(v)| \leq C_\alpha \langle v \rangle^{2s+\gamma-|\alpha|}, \]

where and throughout the paper we use the notation \( \langle \cdot \rangle = \left( 1 + |\cdot|^2 \right)^{1/2} \). The notation \( -\tilde{\triangle} v \) in (1.1) stands for the Fourier multiplier of symbol

\[ |\eta|^{2s} \rho^2(\eta) + |\eta|^{2(1-\rho^2(\eta))}, \]

with \( \rho(\eta) \in C^\infty(\mathbb{R}^n; [0, 1]) \), such that \( \rho = 1 \) if \( |\eta| \geq 2 \) and \( \rho = 0 \) if \( |\eta| \leq 1 \). Here \( \eta \) is the dual variable of \( v \).

Let’s first explain the motivation for studying such a kind of operator \( P \), which is closely linked with the spatially inhomogeneous Boltzmann equation which has singularity in both the kinetic part and the angular part of the cross section. Precisely, non-cutoff Boltzmann equation in \( \mathbb{R}^n \) reads

\[ \partial_t f + v \cdot \partial_x f = Q(f, f), \]

where \( f(t, x, v) \) is a real-valued function, standing for the time-dependent probability density of particles with velocity \( v \) at position \( x \). The right-hand side of (1.5) is the Boltzmann bilinear collision operator which acts only on the velocity variable \( v \) by

\[ Q(f, h)(v) = \int_{\mathbb{R}^n} \int_{S^{n-1}} B(|v - v_\sigma|, \sigma) (f'(v_\sigma' - v) h) \, dv_\sigma \, d\sigma. \]
Here we used the shorthand $f = f(t, x, v), f_\star = f(t, x, v_\star), f' = f(t, x, v')$, and for $\sigma \in \mathbb{S}^{n-1}$, \[ v' = \frac{v + v_\star}{2} + \frac{|v - v_\star|}{2} \sigma, \quad v'_\star = \frac{v + v_\star}{2} - \frac{|v - v_\star|}{2} \sigma. \]

In the above relations, $v'$, $v'_\star$ and $v, v_\star$ are the velocities of a pair of particles before and after collision. The collision cross-section $B(|v - v_\star|, \sigma)$ is a non-negative function which only depends on the relative velocity $|v - v_\star|$ and the deviation angle $\theta$ through $\cos \theta = \frac{v - v_\star}{|v - v_\star|} \cdot \sigma$, and takes the following form
\[ B(|v - v_\star|, \sigma) = \Phi (|v - v_\star|) b (\cos \theta), \quad \cos \theta = \frac{v - v_\star}{|v - v_\star|} \cdot \sigma, \quad 0 \leq \theta \leq \frac{\pi}{2}, \]

where the kinetic part $\Phi$ is given by
\[ \Phi (|v - v_\star|) = |v - v_\star|^\gamma, \quad \gamma > -3, \]

and the angular part $b$ satisfies, with $0 < s < 1$,
\[ b (\cos \theta) \approx \theta^{-(n-1)-2s} \quad \text{as} \quad \theta \rightarrow 0. \]

We refer to [1], [10], [12], [19], [37], [38] and the references therein for the physical background and derivation of the Boltzmann equation, as well as the mathematical theory on the Boltzmann equation. Note that the angular cross-section $b$ is not integrable on the sphere due to the singularity $\theta^{-(n-1)-2s}$, which leads to the conjecture that the nonlinear collision operator should behave like a fractional Laplacian, that is,
\[ Q(f, h) \approx -C_f (|v'|^\gamma (-\Delta_v + \cdots)^s h) + \text{lower order terms}, \]

where $C_f > 0$ is a constant depending only on the physical properties of $f$. Initiated by Desvillettes [15], [16], there have been extensive works which give partial support to the conjecture regarding the smoothness of solutions for the homogeneous Boltzmann equation without angular cutoff, cf. [3], [8], [9], [13], [17], [18], [24], [31], [33]. For the inhomogeneous case the study becomes more complicated, due to the coupling of the transport operator with the collision operator, and the commutator between pseudo-differential operators and the collision operator. Recent works [4]–[7], [20], [21], [30], [34], [35] indicate that the linearized Boltzmann operator around a normalized Maxwellian distribution behaves essentially like the operator
\[ -\langle v' \rangle^\gamma ( - \Delta_v - \Delta_{g_\gamma -1} + v^2)^s h + \text{lower order terms}, \]

where $\Delta_{g_\gamma -1}$ stands for the Laplace-Beltrami operator on the unit sphere $\mathbb{S}^{n-1}$, i.e.,
\[ \Delta_{g_\gamma -1} = \sum_{1 \leq i, j \leq n, \ i \neq j} (v_i \partial_{v_j} - v_j \partial_{v_i})^2. \]

To explain it more precisely, let’s first recall the linearization process. Denote by $\mu$ the normalized Maxwellian distribution; that is
\[ \mu (v) = (2\pi)^{-n/2} e^{-|v|^2/2}. \]

By setting $f = \mu + \sqrt{\mu} h$, we see the perturbation $h$ satisfies the equation
\[ \partial_t h + v \cdot \partial_v h - \mu^{-1/2} Q(\mu, \sqrt{\mu} h) - \mu^{-1/2} Q(\sqrt{\mu} h, \mu) = \mu^{-1/2} Q(\sqrt{\mu} h, \sqrt{\mu} h), \]

since $\partial_t f + v \cdot \partial_v f - Q(f, f) = 0$ and $Q(\mu, \mu) = 0$. Using the notation
\[ \Gamma(f, h) = \mu^{-1/2} Q(\sqrt{\mu} f, \sqrt{\mu} h), \]

we may rewrite the above equation as
\[ \partial_t h + v \cdot \partial_v h - \Gamma(\sqrt{\mu}, h) - \Gamma(h, \sqrt{\mu}) = \Gamma(h, h). \]
With $H^s(\mathbb{R}^n)$ the usual Sobolev space, Alexandre et al. [7] established the following coercivity and upper bound estimates:

$$C^{-1} \left( \| (v)^{1/2} h \|^2_{H^s(\mathbb{R}^n)} + \| (v)^{1/2} h \|^2_{L^2(\mathbb{R}^n)} \right)$$

$$\leq -\Gamma \left( \sqrt{\mu}, h \right) - \Gamma \left( h, \sqrt{\mu} \right) \gamma L_2(\mathbb{R}^n) + \| h \|^2_{L^2(\mathbb{R}^n)}$$

and

$$\left( -\Gamma \left( \sqrt{\mu}, h \right) - \Gamma \left( h, \sqrt{\mu} \right) \gamma \right) L_2(\mathbb{R}^n) \leq C \| (v)^{1/2} h \|^2_{H^s(\mathbb{R}^n)},$$

from which the hypoelliptic properties may be expected for the general spatially inhomogeneous Boltzmann equation without angular cut-off. In order to understand well the linear part $-\Gamma \left( \sqrt{\mu}, h \right) - \Gamma \left( h, \sqrt{\mu} \right)$ of the Boltzmann collision operator, we firstly study the model operator

$$\partial_t + v \cdot \partial_x + a(v)(-\Delta_x)^s + b(v),$$

with $a(v), b(v)$ satisfying the conditions (1.2) and (1.3). We hope this would give insights on the true linearized Boltzmann operator. This motivates the present work on the global hypoellipticity of the operator $P$ given in (1.1).

There have been some related works concerned with the linear model of spatially inhomogeneous Boltzmann equation, which takes the following form

$$P = \partial_t + v \cdot \partial_x - \tilde{a}(t, x, v)(-\Delta_x)^s, \quad \inf_{t, x, v} \tilde{a}(t, x, v) > 0, \quad \tilde{a} \in C^\infty_b,$$

where $C^\infty_b$ stands for the space of smooth functions whose derivatives of any order are bounded. As far as we know, the model operator (1.6) was firstly studied by Morimoto and Xu [32] for $1/3 < s \leq 1$, and then was improved by Chen et al. [14] by virtue of Kohn’s method. Recently Lerner et al. [28] established optimal results using the Wick quantization techniques [25], [26], completing the study of the operator $P$ given in (1.6). However these works are mainly concerned with the local hypoelliptic estimates in the sense that the coefficient $\tilde{a}$ in (1.6) has strictly positive lower bound and bounded derivatives. Compared with the operator in (1.6), the model operator $P$ in (1.1) is closer to the linearized Boltzmann equation in view of the aforementioned coercivity estimate and upper bound estimate. Moreover we do not need the restrictions that $\inf_{t, x, v} \tilde{a}(t, x, v) > 0$ and $\tilde{a} \in C^\infty_b$, since the coefficients in (1.1) may tend to 0 or $+\infty$ as $|v| \to +\infty$, depending on the sign of $\gamma$. In the very recent work of Alexandre [2], a simpler proof was presented following the ideas of Bouchut [11] and Perthame [36], when the coefficient $\tilde{a}(t, x, v)$ in (1.6) has the form $b_0(t_0 + a_0$ with $a_0$ a positive constant, $b_0$ a smooth function and $t_0$ a compactly supported and positive function. We remark that the method used in Alexandre [2] is quite flexible and direct, without using any pseudo-differential calculus, and can be generalized to the operator (1.1) studied here with the similar multiplier choice.

In this paper, we denote by $S$ the Schwartz space of rapidly decreasing functions. Now we state our main result as follows.

**Theorem 1.1** Let $P$ be the operator given in (1.1) with $a(v), b(v)$ satisfying the conditions (1.2) and (1.3). Then there exists a constant $C > 0$ such that for all $f \in S(\mathbb{R}^{2n+1})$ we have

$$\left\| (v)^{1/2} f \right\|_{L^2} + \left\| (v)^{1/2} D_x f \right\|_{L^2} + \left\| a(v) D_x f \right\|_{L^2} + \left\| b(v) f \right\|_{L^2} \leq C \left( \| Pf \|_{L^2} + \| f \|_{L^2} \right),$$

where $\| \cdot \|_{L^2}$ stands for $\| \cdot \|_{L^2(\mathbb{R}^{2n+1})}$, the notation $\langle \cdot \rangle$ stands for $(1 + |\cdot|^2)^{1/2}$, and $D_t = \frac{1}{2} \partial_t, D_x = \frac{1}{2} \partial_x, \text{ etc.}$

**Remark 1.2** It seems that our method can also be applied to the linearized Boltzmann operator $L$ given by

$$Lh = \partial_t h + v \cdot \partial_x h - \Gamma (\sqrt{\mu}, h) - \Gamma (h, \sqrt{\mu}),$$

and give the similar hypoellipticity as above. But the situation is more complicated, and we should pay more attention to handling the commutators between $L$ and pseudo-differential operators. We hope to study this problem in a future work.
Remark 1.3 The estimate in Theorem (1.1) is maximal hypoelliptic estimate, that is

\[ \forall f \in S(\mathbb{R}^{2n+1}), \quad \| (\partial_t + v \cdot \partial_x) f \|_{L^2} + \| a(v)(-\Delta_v)^s f \|_{L^2} + \| b(v) f \|_{L^2} \leq C \left( \| Pf \|_{L^2} + \| f \|_{L^2} \right). \]

We end up the introduction by a few comments on the exponents of derivative terms and weight terms in Theorem 1.1. These exponents seem to be optimal. When restricted to a fixed compact subset \( K \subset \mathbb{R}^{2n+1} \), instead of the whole space, the problems reduce to a local version, and the operator becomes the type given in (1.6), for which the exponent \( 2s/(2s + 1) \) for the regularity in the time and space variables is indeed sharp by using a simple scaling argument (see [28] for more detail). In the particular case when \( s = 1 \), we have a type of differential operator, which seems simpler to handle than the fractional derivatives, and our exponents in the regularity terms and weight terms coincide well with the ones in [22] in the case when \( a(v) \approx \langle v \rangle^\gamma \).

2 Notations and some symbolic calculus

2.1 Notations and some basic facts on symbolic calculus

Notice that the diffusion term in (1.1) is an operator only with respect to the velocity variable \( v \). So it is convenient to take partial Fourier transform in the \( t, x \) variables, and then to study the operator on the Fourier side

\[ \hat{\mathcal{P}} = i(\tau + v \cdot \xi) + a(v)(D_v)^{2s} + b(v), \quad 0 < s \leq 1, \]

where and throughout the paper, \((\tau, \xi)\) always stand for the dual variables of \((t, x)\) and are considered as parameters, while \( \eta \) will be used to denote the dual variable of \( v \). Note that the diffusion term is not exactly the same as the one in (1.1), and we write \((D_v)^{2s}\) instead of \((-\Delta_v)^s\) just for simplifying the notation. This brings no essential difference in the point of \( L^2 \) estimate, because we have (see Remark 3.3 below)

\[ \| a(v)(D_v)^{2s} f - a(v)(-\Delta_v)^s f \|_{L^2} \lesssim \| (v)^{2s+\gamma} f \|_{L^2} + \| (v)^{s+\gamma} (D_v)^sf \|_{L^2} \lesssim \| Pf \|_{L^2} + \| f \|_{L^2}. \]

Since our analysis is on \( \mathbb{R}^n \), we will use \((\cdot, \cdot)_{L^2} \) and \( \| \cdot \|_{L^2} \), instead of \((\cdot, \cdot)_{L^2(\mathbb{R}^n)} \) and \( \| \cdot \|_{L^2(\mathbb{R}^n)} \), to denote the inner product and norm in \( L^2(\mathbb{R}^n) \), if no confusion occurs.

To simplify the notation, by \( A \lesssim B \) we mean there exists a positive constant \( C \) independent of the parameters \((\tau, \xi)\), such that \( A \leq C B \), and similarly for \( A \gtrsim B \). While the notation \( A \approx B \) means both \( A \lesssim B \) and \( B \lesssim A \) hold. We will use \( h^\prime \) and \( h^{\prime \prime} \) to stand for, respectively, the first and the second derivative of a given function \( h \).

Now we introduce some notations of phase space analysis and recall some basic properties of symbolic calculus, and refer to [23] and [27] for detailed discussions. Let \( p \in \mathcal{S}^\prime(\mathbb{R}^n \times \mathbb{R}^n) \) be a tempered distribution and let \( l \in \mathbb{R} \). Denoting by \( S^\prime(\mathbb{R}^n) \) the dual of \( \mathcal{S}(\mathbb{R}^n) \), we define the operator \( \text{op}_l p : \mathcal{S}(\mathbb{R}^n) \rightarrow S^\prime(\mathbb{R}^n) \) by the formula

\[ \langle (\text{op}_l p)(f), \eta \rangle_{S^\prime, S} = \langle p, \Omega_{f, h}(l) \rangle_{\mathcal{S}, \mathcal{S}}, \]

where

\[ \Omega_{f, h}(l)(v, \eta) = \int e^{-2i\pi \eta \cdot v} f(v + (1 - l)z)h(v - lz) \, dz. \]

In particular we denote by \( p^w \) the so-called Weyl quantization of \( p \), i.e.,

\[ p^w = \text{op}_{1/2} p, \]

and denote

\[ p(v, D_v) = \text{op}_0 p. \]

Let’s recall the definitions of admissible metric and admissible weight (see [23] and [27] for instance). Suppose for each \( Z \in \mathbb{R}^d \), \( g_Z \) is a positive-definite quadratic form on \( \mathbb{R}^d \). We say \( g \) is an admissible metric if the following properties are fulfilled:

\[ \sum_{i,j} g_{ij} y_i y_j \geq C \sum_i y_i^2, \quad \text{for all } y \in \mathbb{R}^d, \]

\[ \sum_{i,j} g_{ij} y_i y_j \leq C \sum_i y_i^2, \quad \text{for all } y \in \mathbb{R}^d. \]
(i) (slowly varying condition) there exists a positive number \( r > 0 \) such that for any \( X, Y \in \mathbb{R}^d \) and any \( T \in \mathbb{R}^d \), if \( g_X(X - Y) \leq r \) then \( g_X(T) \approx g_Y(T) \);

(ii) (uncertainty principle) for each \( X \in \mathbb{R}^d \) we have \( g_X \leq g_X^\sigma \), where \( g_X^\sigma = \sigma \cdot g_X \sigma \) with \( \sigma = \begin{pmatrix} 0 & \text{Id} \\ -\text{Id} & 0 \end{pmatrix} \);

(iii) (temperateness) there exists a number \( N \geq 0 \) such that for any \( X, Y \in \mathbb{R}^d \) and any \( T \in \mathbb{R}^d \) we have

\[
\frac{g_X(T)}{g_Y(T)} \lesssim (1 + g_X^\sigma(X - Y))^N.
\]

A weight function \( m : \mathbb{R}^d \to [0, +\infty[ \) is called admissible for an admissible metric \( g \) if the following properties are fulfilled:

(I) there exists a positive number \( \tilde{r} > 0 \) such that for any \( X, Y \in \mathbb{R}^d \), if \( g_X(X - Y) \leq \tilde{r} \) then \( m(X) \approx m(Y) \);

(II) there exists a number \( \tilde{N} \geq 0 \) such that for any \( X, Y \in \mathbb{R}^d \) we have

\[
m(X) \approx m(Y) \lesssim (1 + g_X^\sigma(X - Y))^\tilde{N}.
\]

Throughout the paper we always let \( g \) be an admissible metric satisfying the additional condition that \( g_{e,\alpha}(\bar{v}, \bar{\eta}) = g_{e,\alpha}(\bar{v}, -\bar{\eta}) \) for all \((v, \eta), (\bar{v}, -\bar{\eta}) \in \mathbb{R}^d \), and let \( m \) be an admissible weight for \( g \). Consider a symbol \( p(\tau, \xi, v, \eta) \) as a function of \((v, \eta)\) with parameters \((\tau, \xi)\), and we say \( p \in S(m, g) \) uniformly with respect to \((\tau, \xi)\), if for any \( k \in \mathbb{N} \) and any \( X, T_1, \ldots, T_k \in \mathbb{R}^d \) one has

\[
\left| \langle p^{(k)}(\tau, \xi, X, T_1 \otimes \cdots \otimes T_k) \rangle \right| \leq C_k m(X) \left( g_X(T_1)^{1/2} g_X(T_2)^{1/2} \cdots g_X(T_k)^{1/2} \right),
\]

with \( C_k \) a constant depending only on \( k \), but independent of \((\tau, \xi)\). To simplify the notations, we will omit the parameters \((\tau, \xi)\) in symbols, and by \( p \in S(m, g) \) we always mean that \( p \) satisfies the above inequality uniformly with respect to \((\tau, \xi)\). In the paper we will use frequently the \( L^2 \) continuity theorem in the class \( S(1, g) \), which says that for \( p \in S(1, g) \) and \( l \in \mathbb{R} \), we have

\[
\forall u \in L^2, \quad \| (\text{op}_l p) u \|_{L^2} \lesssim \| u \|_{L^2}.
\]

Given \( p \in S(m, g) \), we have the formula of changing quantization (see Proposition 1.1.10 and Theorem 2.3.18 of [27]):

\[
\forall l \in \mathbb{R}, \quad \text{op}_l p = \text{op}_0 (J^l p) = (J^l p)(v, D_v),
\]

where \( J^l p \) has the expansion

\[
J^l p = \sum_{0 \leq j < k} \frac{l^j}{j!} (2i\pi D_v \cdot D_\eta)^j p \in S(m^{1/k}, g),
\]

with \( \lambda^{1/k} \) the Planck’s constant with respect to the metric \( g \), defined by

\[
\lambda(X) = \inf_T \left( g_X^\sigma(T)/g_X(T) \right)^{1/2}.
\]

If \( p_i \in S(m_i, g), i = 1, 2 \), then we have (e.g. Theorem 2.3.7 and Theorem 2.3.19 of [27])

\[
p_{1^w} p_{2^w} = (p_1 \xi p_2)^w \in \text{op}_{1/2} (S(m_1 m_2, g)),
\]

and

\[
p_{1}(v, D_v)p_{2}(v, D_v) = (p_1 \circ p_2)(v, D_v) \quad \text{with} \quad p_1 \circ p_2 \in S(m_1 m_2, g).
\]

Finally let’s recall some basic properties of the Wick quantization, and refer the reader to the works of Lerner [25]–[27] for thorough and extensive presentations of this quantization and some of its applications. Given
\[ X = (v, \eta) \] be a point in \( \mathbb{R}^{2n} \), we define the wave-packets transform of a function \( u \in S(\mathbb{R}^n) \) by

\[ W u (X) = (u, \varphi_X)_{L^2(\mathbb{R}^n)} = 2^{n/4} \int_{\mathbb{R}^n} u(z) e^{-\pi |z-v|^2} e^{2i \pi (z-v) \eta} \, dz, \]

where

\[ \varphi_X(z) = 2^{n/4} e^{-\pi |z-v|^2} e^{2i \pi (z-v) \eta}, \quad z \in \mathbb{R}^n. \]

We remark that \( W \) is an isometric mapping from \( L^2(\mathbb{R}^n) \) to \( L^2(\mathbb{R}^{2n}) \) with adjoint \( W^* \). The Wick quantization of any \( L^\infty \) symbol \( p \) is defined as follows:

\[ p^{\text{Wick}} = W^* p W. \]

Thus

\[ p(v, \eta) \geq 0 \quad \text{for all} \quad (v, \eta) \in \mathbb{R}^{2n} \quad \text{implies} \quad p^{\text{Wick}} \geq 0. \]

The Wick and Weyl quantizations of a symbol \( p \) are linked by the following identities (see Proposition 2.4.3 in [27]):

\[ p^{\text{Wick}} = p^w + r^w \]

with

\[ r(X) = \int_0^1 \int_{\mathbb{R}^n} (1 - \theta) p''(X + \theta Y) Y^2 e^{-2\pi |Y|^2} 2^n \, dY d\theta. \]

Throughout the paper, the Poisson bracket \( \{p, q\} \) is defined by

\[ \{p, q\} = \frac{\partial p}{\partial \eta} \cdot \frac{\partial q}{\partial v} - \frac{\partial p}{\partial v} \cdot \frac{\partial q}{\partial \eta}. \]

In view of the proof of Proposition 3.4 in [25], one has

\[ p^{\text{Wick}, q}^{\text{Wick}} = \left[ pq - \frac{1}{4\pi} p' \cdot q' + \frac{1}{4i\pi} \{p, q\} \right]^{\text{Wick}} + T, \]

with \( T \) a bounded operator in \( L^2(\mathbb{R}^{2n}) \), when \( p \in L^\infty(\mathbb{R}^{2n}) \) and \( q \) is a smooth symbol whose derivatives of order \( \geq 2 \) are bounded on \( \mathbb{R}^{2n} \).

### 2.2 Global pseudo-differential calculus

We begin with some facts on symbolic calculus which will be used frequently. Observe that

\[ \langle v \rangle^\ell \in S \big( \langle v \rangle^\ell, \langle v \rangle^{-2} |dv|^2 + \langle \eta \rangle^{-2} |d\eta|^2 \big), \quad \langle \eta \rangle^\ell \in S \big( \langle \eta \rangle^\ell, \langle v \rangle^{-2} |dv|^2 + \langle \eta \rangle^{-2} |d\eta|^2 \big), \]

and \( a(v), b(v) \in S \big( \langle v \rangle^{2r+\gamma}, \langle v \rangle^{-2} |dv|^2 + \langle \eta \rangle^{-2} |d\eta|^2 \big) \) due to (1.3). Then symbolic calculus (see Theorem 2.3.19 of [27] for instance or the Appendix) shows that for any \( \delta \in \mathbb{R} \) and any \( \kappa \in \mathbb{R} \) we have

\[ \left[ (D_v)^\delta, \langle v \rangle^\ell \right] \in \mathcal{O}_0 \left( S \left( \langle v \rangle^{\delta-1} \langle \eta \rangle^{\delta-1}, \langle v \rangle^{-2} |dv|^2 + \langle \eta \rangle^{-2} |d\eta|^2 \right) \right), \]

and

\[ \left[ (D_v)^\delta, a \right] \left[ (D_v)^\delta, b \right] \in \mathcal{O}_0 \left( S \left( \langle v \rangle^{2r+\gamma-1} \langle \eta \rangle^{\delta-1}, \langle v \rangle^{-2} |dv|^2 + \langle \eta \rangle^{-2} |d\eta|^2 \right) \right), \]

where and throughout the paper \( [A, B] \) stands for the commutator between \( A \) and \( B \) defined by \([A, B] = AB - BA\).

**Lemma 2.1** Let \( a(v) \) be the coefficient given in (1.1), with the conditions (1.2) and (1.3) fulfilled. Then

\[ \forall \rho \in \mathbb{R}, \forall |\alpha| \geq 0, \quad \left| \partial^\alpha v(a^\rho) \right| \lesssim a^\rho. \]

Moreover we have

\[ \forall \rho \leq 1, \forall |\alpha| \geq 2, \quad \left| \partial^\alpha v(a^\rho) \right| \lesssim \langle v \rangle^{\gamma \rho}. \]
and
\[ \forall \rho \leq 1, \forall |\alpha| \geq 2, \quad |\partial^\alpha (a^\rho / (v)^\gamma)| \leq C_\alpha, \]
where \( \gamma \) is the number given in (1.2).

\[ \text{Proof.} \] The proof is divided into four steps.

**Step 1:** Let’s firstly prove that
\[ \forall |\alpha| \geq 0, \forall v \in \mathbb{R}^n, \quad |\partial^\alpha a(v)| \lesssim a(v). \]
To do so, we choose a smooth non-negative function \( h(v) : \mathbb{R}^n \to \mathbb{R}_+ \) which is defined by
\[ h(v) = a(v) / (v)^\gamma. \]

In view of (1.2) and (1.3), it follows from direct computation that \( h''(v) \in L^\infty(\mathbb{R}^n) \). Then the classical inequality for non-negative functions (see for instance Lemma 4.3.8 in [27]) yields
\[ \left| h'(v) \right|^2 \leq 2h(v) \left\| h'' \right\|_{L^\infty} \lesssim \frac{a(v)}{(v)^\gamma}. \]

This along with the fact that
\[ \frac{|a'(v)|^2}{(v)^{2\gamma}} \leq 2 \left| h'(v) \right|^2 + 2 |\gamma|^2 \frac{|a(v)|^2}{(v)^{2\gamma+2}} \]
gives
\[ \left| a'(v) \right|^2 \lesssim \frac{a(v)}{(v)^\gamma} + \frac{|a(v)|^2}{(v)^{2\gamma+2}} \lesssim \frac{a(v)}{(v)^\gamma}, \]
where the last inequality holds because by (1.2) we have \( \frac{a(v)}{(v)^{2\gamma+2}} \lesssim \frac{1}{(v)^\gamma} \). Consequently,
\[ \forall v \in \mathbb{R}^n, \quad |a'(v)|^2 \lesssim a(v)(v)^\gamma, \]
which together with the fact that \( (v) \gamma \lesssim a(v) \) due to (1.2), gives
\[ \forall v \in \mathbb{R}^n, \quad |a'(v)| \lesssim a(v). \]

Furthermore using (1.2) and (1.3), we could verify that
\[ \forall |\alpha| \geq 2, \forall v \in \mathbb{R}^n, \quad |\partial^\alpha a(v)| \lesssim a(v). \]

By the above two estimates we get (2.15).

**Step 2:** Next let’s show that for any non-negative integer \( k \in \mathbb{Z}_+ \) one has
\[ (\mathcal{A}_k) : \forall |\alpha| \leq k, \forall \rho \in \mathbb{R}, \quad |\partial^\alpha (a^\rho)| \lesssim a^\rho. \]

We will use induction on \( k \) to prove the above conclusion. It is clear that \( (\mathcal{A}_0) \) holds. Now assuming \( (\mathcal{A}_k) \) holds for some non-negative integer \( k \), one has to show the validity of \( (\mathcal{A}_{k+1}) \). For any \( |\alpha| \leq k + 1 \) we may choose \( \tilde{\alpha} \) such that \( |\tilde{\alpha}| = |\alpha| - 1 \). So \( |\tilde{\alpha}| \leq k \) and
\[ \forall \rho \in \mathbb{R}, \quad \partial^\alpha (a^\rho) = \partial^\alpha_\rho (\rho a^{\rho-1} a') = \rho \sum_{|\beta| \leq |\tilde{\alpha}|} C^\rho_\beta (\partial^\beta a^{\rho-1}) (\partial^{\tilde{\alpha} - \beta} a'), \]
the last equality using the Leibniz formula. The validity of \( (\mathcal{A}_k) \) implies
\[ \forall |\beta| \leq k, \quad |\partial^\beta a^{\rho-1}| \lesssim a^{\rho-1}. \]
Moreover by (2.15), we have \( |\partial^{\tilde{\alpha} - \beta} a'| \lesssim a \). Thus
\[ \forall |\alpha| \leq k + 1, \forall \rho \in \mathbb{R}, \quad |\partial^\alpha (a^\rho)| \lesssim a^{\rho-1} a = a^\rho, \]
yielding the validity of \( (\mathcal{A}_{k+1}) \). Then the conclusion (2.17) follows. We have proven (2.12).
Step 3: In this step we will prove (2.13), i.e.,
\[ \forall \rho \leq 1, \forall |\alpha| \geq 2, \quad |\partial^\alpha (a^\rho)| \lesssim \langle v \rangle^{\gamma \rho}. \] (2.18)

Given any \( \alpha \) with \( |\alpha| \geq 2 \), we may choose \( \tilde{\alpha} \) such that \( |\tilde{\alpha}| = |\alpha| - 2 \). Then Leibniz formula yields
\[
\partial^\alpha (a^\rho) = \tilde{\partial}^{\tilde{\alpha}} \left( \rho a^{\rho-1} a^n + \rho (\rho - 1) a^{\rho-2} a' a' \right)
\]
\[ = \rho \sum_{\rho \leq \tilde{\alpha}} C^\rho_{\tilde{\alpha}} (\partial^{\tilde{\alpha}} a^{\rho-1}) (\partial^{\rho} a^n)
\]
\[ + \rho (\rho - 1) \sum_{\beta \leq \rho} C^\rho_{\tilde{\alpha}} C^\beta_{\rho} (\partial^{\tilde{\alpha}} a^{\rho-2}) (\partial^{\beta} a'). \]

Moreover, by (2.17), (1.2) and (1.3), we have
\[ |(\partial^{\tilde{\alpha}} a^{\rho-1}) (\partial^{\rho} a^n)| \lesssim a^{\rho-1} \langle v \rangle^\gamma \]
and
\[
|(\partial^{\tilde{\alpha}} a^{\rho-2}) (\partial^{\beta} a')(\partial^{\rho} a^n)| \lesssim a^{\rho-2} \left( |a'|^2 + \langle v \rangle^\gamma |a'| + \langle v \rangle^\gamma \langle v \rangle^\gamma \right)
\]
\[ \lesssim a^{\rho-1} \langle v \rangle^\gamma, \]
where the last inequality follows from (2.16) and the fact that \( |a'| + \langle v \rangle^\gamma \lesssim a \) due to (2.15) and (1.2). As a result one has
\[ \forall \rho \leq 1, \forall |\alpha| \geq 2, \quad |\partial^\alpha (a^\rho)| \lesssim C a a^{\rho-1} \langle v \rangle^\gamma \lesssim \langle v \rangle^{\gamma \rho}. \]

the last inequality using (1.2). The conclusion (2.13) follows.

Step 4: It remains to prove (2.14), which is a consequence of (2.16) and (2.13). In fact by Leibniz formula we have, with \( |\alpha| \geq 2 \),
\[
\partial^\alpha (a^\rho / \langle v \rangle^\gamma) = a^n \partial^\alpha (\langle v \rangle^{-\gamma}) + \sum_{|\beta| = 1} C^\rho_{\tilde{\alpha}} (\partial^{\tilde{\alpha}} (\langle v \rangle^{-\gamma})) (\partial^{\rho} a^n)
\]
\[ + \sum_{2 \leq |\beta| \leq |\alpha|} C^\rho_{\tilde{\alpha}} C^\beta_{\rho} (\partial^{\tilde{\alpha}} (\langle v \rangle^{-\gamma})) (\partial^{\beta} a^n). \]

It follows from (1.2) that, with \( |\alpha| \geq 2 \) and \( \rho \leq 1 \),
\[ |a^n \partial^\alpha (\langle v \rangle^{-\gamma})| \lesssim \left( \langle v \rangle^{\rho \gamma} + \langle v \rangle^{(2+\gamma)\rho} \right) \langle v \rangle^{-\gamma \rho - 2} \lesssim 1. \]
Moreover by (2.16) we have, with \( |\beta| = 1, |\alpha| \geq 2 \) and \( \rho \leq 1 \),
\[
\left( \partial^{\tilde{\alpha}} (\langle v \rangle^{-\gamma}) \right) (\partial^{\rho} a^n) \lesssim \langle v \rangle^{-\gamma - |v| - \rho} a^{\rho-1} a^{1/2} (\langle v \rangle^{\gamma})^2
\]
\[ \lesssim \langle v \rangle^{-\gamma - 1} \langle v \rangle^{(\rho-1)\gamma} \langle v \rangle^{\gamma+\gamma/2} \langle v \rangle^{\gamma/2} \lesssim 1. \]

Finally, for any \( |\beta| \geq 2 \) we have
\[ \left( \partial^{\tilde{\alpha}} (\langle v \rangle^{-\gamma}) \right) (\partial^{\beta} a^n) \right) \lesssim \langle v \rangle^{-\gamma} \langle v \rangle^{\gamma} \lesssim 1 \]
due to (2.13). These inequalities yield (2.14), completing the proof. \( \square \)

Lemma 2.2 Let \( \rho \leq 1 \) and \( \delta \in \mathbb{R} \) be given. Then

(i) for \( q \in S \left( \langle \eta \rangle^4, |dv|^2 + \langle \eta \rangle^{-2} |d\eta|^2 \right) \) given, we have
\[ \left[ q(v, \partial^\alpha) \right] = a^{\rho-1} (v) R(v, \partial^\alpha) + r(v, \partial^\alpha), \] (2.19)
with \( R \in S \left( \langle v \rangle^{\gamma+\gamma}, |dv|^2 + |d\eta|^2 \right) \) and \( r \in S \left( \langle v \rangle^{\gamma} \langle \eta \rangle^{-2}, |dv|^2 + |d\eta|^2 \right); \)
(ii) for $\omega \in S \left(1, |dv|^2 + |d\eta|^2 \right)$ given, we have
\[
\forall f \in S(\mathbb{R}^n), \quad \left\| [\omega(v, D_v), a^\rho] f \right\|_{L^2} \lesssim \left\| a^\rho f \right\|_{L^2} + \left\| \langle v \rangle^{\gamma \rho} f \right\|_{L^2}.
\] (2.20)

**Proof.** We may write
\[
[q(v, D_v), a^\rho] = [q(v, D_v), a^\rho / \langle v \rangle^{\gamma \rho}] \langle v \rangle^{\gamma \rho} + a^\rho / \langle v \rangle^{\gamma \rho} \left[q(v, D_v), \langle v \rangle^{\gamma \rho} \right].
\]
From Theorem 1.1.20 in [27] (see Theorem A.2 in the Appendix), it follows that the symbol of the commutator
\[
[q(v, D_v), a^\rho / \langle v \rangle^{\gamma \rho}]
\]
is
\[
\frac{\partial_n (q(v, \eta)) \cdot \partial_v (a^\rho / \langle v \rangle^{\gamma \rho})}{2i\pi} + \tilde{r}(v, \eta),
\]
with
\[
\tilde{r}(v, \eta) = \int_0^1 (1 - \theta)^2 e^{2i\pi \theta D_v \cdot D_\xi} (2i\pi D_v \cdot D_\xi)^2 q(v, \xi) \left\{ a^\rho (z) \left( \frac{z}{\langle \chi \rangle^{\gamma \rho}} \right) \right\} d\theta \bigg|_{\gamma \xi = \eta}.
\]
Note that
\[
(\gamma, \xi) \longrightarrow (2i\pi D_v \cdot D_\xi)^2 q(v, \xi) \left\{ a^\rho (z) \left( \frac{z}{\langle \chi \rangle^{\gamma \rho}} \right) \right\} \in S \left(\langle \xi \rangle^{\delta - 2}, |d\xi|^{2} + |\gamma \xi|^{2} \right)
\]
due to the estimate (2.14). Then we conclude, by Lemma 4.1.5 in [27] (see Proposition A.3 in the Appendix),
\[
\tilde{r}(v, \eta) \in S \left(\langle \eta \rangle^{\delta - 2}, |dv|^{2} + |d\eta|^{2} \right) \subset S \left(\langle \eta \rangle^{\delta - 2}, |dv|^{2} + |d\eta|^{2} \right).
\]
Thus
\[
[q(v, D_v), a^\rho / \langle v \rangle^{\gamma \rho}] (v)^{\gamma \rho} = \left(a^\rho / \langle v \rangle^{\gamma \rho}\right)^{\prime} \cdot op_0 \left((2i\pi)^{-1} \partial_n (q(v, \eta))\right) (v)^{\gamma \rho} + \tilde{r}(v, D_v) (v)^{\gamma \rho}
\]
\[
= a^{\gamma \rho - 1} R_1 (v, D_v) + r(v, D_v),
\]
where
\[
r(v, D_v) = \tilde{r}(v, D_v) (v)^{\gamma \rho} \in op_0 \left(S \left(\langle v \rangle^{\gamma \rho} \langle \eta \rangle^{\delta - 1}, |dv|^{2} + |d\eta|^{2} \right) \right)
\]
and
\[
R_1 (v, D_v) = a^{\gamma \rho - 1} \left(a^\rho / \langle v \rangle^{\gamma \rho}\right)^{\prime} \cdot op_0 \left((2i\pi)^{-1} \partial_n (q(v, \eta))\right) (v)^{\gamma \rho},
\]
which belongs to \( op_0 \left(S \left(\langle v \rangle^{\gamma \rho + \gamma \rho} \langle \eta \rangle^{\delta - 1}, |dv|^{2} + |d\eta|^{2} \right) \right) \) since
\[
a^{\gamma \rho - 1} \left(a^\rho / \langle v \rangle^{\gamma \rho}\right)^{\prime} = \rho a^\rho / \langle v \rangle^{\gamma \rho} + a(v) \left(\langle v \rangle^{-\gamma \rho}\right)^{\prime} \in op_0 \left(S \left(\langle v \rangle^{\gamma \rho - \gamma \rho} \langle \eta \rangle^{\delta - 1}, |dv|^{2} + |d\eta|^{2} \right) \right)
\]
due to (1.3). On the other hand, applying Theorem 2.3.8 in [27] (see Theorem A.4 in the Appendix), we have
\[
[q(v, D_v), (v)^{\gamma \rho}] = \left\langle (J^{\gamma \rho} q(v, \eta))^{\prime}, (v)^{\gamma \rho} \right\rangle \in S \left(\langle v \rangle^{\gamma \rho - 1} \langle \eta \rangle^{\delta - 1}, |dv|^{2} + |d\eta|^{2} \right),
\]
and thus
\[
a^\rho / \langle v \rangle^{\gamma \rho} [q(v, D_v), (v)^{\gamma \rho}] = a^{\gamma \rho - 1} R_2 (v, D_v)
\]
with
\[
R_2 (v, D_v) = a / \langle v \rangle^{\gamma \rho} \left[q(v, D_v), (v)^{\gamma \rho} \right] \in op_0 \left(S \left(\langle v \rangle^{\gamma \rho + \gamma \rho} \langle \eta \rangle^{\delta - 1}, |dv|^{2} + |d\eta|^{2} \right) \right)
\]
due to (1.3). Letting \( R = R_2 + R_1 \), we have proven (2.19).

To obtain (2.20), we write
\[
[\omega(v, D_v), a^\rho] = [\omega(v, D_v), (v)^{\gamma \rho}] a^\rho / \langle v \rangle^{\gamma \rho} + (v)^{\gamma \rho} [\omega(v, D_v), a^\rho / \langle v \rangle^{\gamma \rho}],
\]
This gives
\[
\left\| [\omega(v, D_v), a^\rho] f \right\|_{L^2} \lesssim \left\| [\omega(v, D_v), (v)^{\gamma \rho}] a^\rho / \langle v \rangle^{\gamma \rho} f \right\|_{L^2} + \left\| (v)^{\gamma \rho} [\omega(v, D_v), a^\rho / \langle v \rangle^{\gamma \rho}] f \right\|_{L^2}
\]
\[
\lesssim \left\| a^\rho f \right\|_{L^2} + \left\| (v)^{\gamma \rho} [\omega(v, D_v), a^\rho / \langle v \rangle^{\gamma \rho}] f \right\|_{L^2},
\]
where the last inequality holds because
\[
\left[ \omega(v, D_v), \langle v \rangle^{\gamma \rho} \right] \langle v \rangle^{-\gamma \rho} \in \text{op}_0 \left( S \left( 1, |d v|^2 + |d \eta|^2 \right) \right).
\]

On the other hand, by the similar arguments as above, we have, with \( \tilde{\omega} = (2i \pi)^{-1} \partial_\omega \omega(v, \eta) \),
\[
\left[ \omega(v, D_v), \frac{a^\rho}{\langle v \rangle^{\gamma \rho}} \right] = (a^\rho / \langle v \rangle^{\gamma \rho})' \tilde{\omega}(v, D_v) + r_1(v, D_v)
\]
\[
= \tilde{\omega}(v, D_v) (a^\rho / \langle v \rangle^{\gamma \rho})' + \left( [a^\rho / \langle v \rangle^{\gamma \rho}]' - (a^\rho / \langle v \rangle^{\gamma \rho})' \tilde{\omega}(v, D_v) \right) + r_1(v, D_v)
\]
\[
= \tilde{\omega}(v, D_v) (a^\rho / \langle v \rangle^{\gamma \rho})' + \frac{(a^\rho / \langle v \rangle^{\gamma \rho})' (\partial_\omega \tilde{\omega})(v, D_v)}{2i \pi}
\]
\[
+ r_2(v, D_v) + r_1(v, D_v),
\]
where \( r_1(v, D_v), r_2(v, D_v) \in \text{op}_0 \left( S \left( 1, |d v|^2 + |d \eta|^2 \right) \right) \). Moreover note that
\[
\frac{(a^\rho / \langle v \rangle^{\gamma \rho})' (\partial_\omega \tilde{\omega})(v, D_v)}{2i \pi} \in \text{op}_0 \left( S \left( 1, |d v|^2 + |d \eta|^2 \right) \right)
\]
due to (2.14). Then we may write
\[
\langle v \rangle^{\gamma \rho} \left[ \omega(v, D_v), \frac{a^\rho}{\langle v \rangle^{\gamma \rho}} \right] = p_1(v, D_v) (a^\rho / \langle v \rangle^{\gamma \rho})' + p_2(v, D_v) / \langle v \rangle^{\gamma \rho},
\]
where
\[
p_1 = \langle v \rangle^{\gamma \rho} \tilde{\omega}(v, D_v) / \langle v \rangle^{-\gamma \rho} \in \text{op}_0 \left( S \left( 1, |d v|^2 + |d \eta|^2 \right) \right)
\]
and
\[
p_2 = \langle v \rangle^{\gamma \rho} \left( \frac{(a^\rho / \langle v \rangle^{\gamma \rho})' (\partial_\omega \tilde{\omega})(v, D_v)}{2i \pi} + r_1(v, D_v) + r_2(v, D_v) \right) / \langle v \rangle^{-\gamma \rho} \in \text{op}_0 \left( S \left( 1, |d v|^2 + |d \eta|^2 \right) \right).
\]
This, along with the \( L^2 \) continuity theorem in the class \( S \left( 1, |d v|^2 + |d \eta|^2 \right) \), gives
\[
\| \langle v \rangle^{\gamma \rho} \left[ \omega(v, D_v), \frac{a^\rho}{\langle v \rangle^{\gamma \rho}} \right] f \|_{L^2} \lesssim \| (a^\rho / \langle v \rangle^{\gamma \rho})' f \|_{L^2} + \| (a^\rho) f \|_{L^2},
\]
the last inequality holding because
\[
\| (a^\rho / \langle v \rangle^{\gamma \rho})' f \|_{L^2} \lesssim a^\rho
\]
due to (2.12). We have proven (2.20), completing the proof. \( \square \)

**Lemma 2.3** (Changing the quantization)

(i) Let \( 2s \in \mathbb{R} \) be given. Then we have
\[
\left( a(v) \langle \eta \rangle^{2s} \right)_{\text{Wick}} = a(v) \langle D_v \rangle^{2s} + R_1(v, D_v) + R_2(v, D_v) + R_3(v, D_v),
\]
with
\[
R_1(v, \eta) \in S \left( \langle v \rangle^\gamma \langle \eta \rangle^{2s}, |d v|^2 + |d \eta|^2 \right),
\]
\[
R_2(v, \eta) \in S \left( \langle v \rangle^{s+\gamma} \langle \eta \rangle^s, |d v|^2 + |d \eta|^2 \right),
\]
\[
R_3(v, \eta) \in S \left( \langle v \rangle^{2s+\gamma}, |d v|^2 + |d \eta|^2 \right).
\]

(ii) We have
\[
\forall \rho \leq 1, \quad \left( a^\rho \right)_{\text{Wick}} = a^\rho + r(v, D_v),
\]
with \( r(v, \eta) \in S \left( \langle v \rangle^{\gamma \rho}, |d v|^2 + |d \eta|^2 \right). \)
Let $N \in \mathbb{N}$ be given. Then
\[ \forall \rho \in \mathbb{R}, \quad \left((N + |v|^2)^{\rho/2}\right)^{\text{Wick}} = (N + |v|^2)^{\rho/2} + r_N(v, D_v), \tag{2.23} \]
with $r_N(v, \eta) \in S \left((N + |v|^2)^{(\rho-2)/2}, |dv|^2 + |d\eta|^2\right)$ uniformly with respect to $N$.

**Proof.** We have, by (2.7),
\[ (a(v) \langle \eta \rangle^{2\gamma})^{\text{Wick}} = (a(v) \langle \eta \rangle^{2\gamma})^w + R^w, \]
where
\[
R(X) = \int_0^1 \int_{\mathbb{R}^n} (1 - \theta) F''(X + \theta Y) e^{-2\pi |Y|^2} 2^n dY d\theta \quad \text{(with } F = a(v) \langle \eta \rangle^{2\gamma}, \ Y = (\bar{v}, \bar{\eta}) )
\]
\[
= \int_0^1 \int_{\mathbb{R}^n} (1 - \theta) a''(v + \theta \bar{v}) \langle \eta \rangle^{2\gamma} |\bar{v}|^2 e^{-2\pi |\bar{v}|^2} 2^n dY d\theta
\]
\[
+ \int_0^1 \int_{\mathbb{R}^n} (1 - \theta) a'(v + \theta \bar{v}) G'(\eta + \theta \bar{\eta}) \bar{v} \bar{\eta} e^{-2\pi (|\bar{v}|^2 + |\bar{\eta}|^2)} 2^n dY d\theta \quad \text{(with } G = \langle \eta \rangle^{2\gamma})
\]
\[
= \tilde{R}_1 + \tilde{R}_2 + \tilde{R}_3.
\]

It follows from (1.2) that
\[
|\partial^a \partial^b \tilde{R}_1| \lesssim \int_0^1 \int_{\mathbb{R}^n} (v + \theta \bar{v})^\gamma \langle \eta \rangle^{2\gamma} |\bar{v}|^2 e^{-2\pi |\bar{v}|^2} 2^n d\bar{v} d\theta
\]
\[
\lesssim \langle v \rangle^\gamma \langle \eta \rangle^{2\gamma} \int_{\mathbb{R}^n} |\bar{v}|^{|\gamma|} |\bar{v}|^{\gamma} e^{-2\pi |\bar{v}|^2} 2^n d\bar{v}
\]
\[
\lesssim \langle \eta \rangle^{2\gamma},
\]
which yields
\[ \tilde{R}_1 \in S \left((\langle v \rangle + |dv|^2 + |d\eta|^2\right). \]

Similarly we have, by virtue of (1.3),
\[ \tilde{R}_2 \in S \left((\langle v \rangle + |dv|^2 + |d\eta|^2\right), \]
and
\[ \tilde{R}_3 \in S \left((\langle v \rangle + |dv|^2 + |d\eta|^2\right). \]

On the other hand, we have, by (2.2),
\[
(a(v) \langle \eta \rangle^{2\gamma})^w = \text{op}_0 \left(J^{1/2} \left(a(v) \langle \eta \rangle^{2\gamma}\right)\right)
\]
\[
= a(v) \langle D_v \rangle^{2\gamma} + \text{op}_0 \left(J^{1/2} \left(a(v) \langle \eta \rangle^{2\gamma}\right) - a(v) \langle \eta \rangle^{2\gamma}\right),
\]
with
\[
J^{1/2} \left(a(v) \langle \eta \rangle^{2\gamma}\right) - a(v) \langle \eta \rangle^{2\gamma} \in S \left((\langle v \rangle + |dv|^2 + |\eta|^{\gamma} + |d\eta|^2\right)
\]
due to (2.3), since
\[ a(v) \langle \eta \rangle^{2\gamma} \in S \left((\langle v \rangle + |dv|^2 + |\eta|^{\gamma} + |d\eta|^2\right). \]

Then the desired estimate (2.21) follows if we choose $R_j(v, \eta) = J^{1/2} \tilde{R}_j(v, \eta)$ with $j = 1, 3$, and
\[
R_2(v, \eta) = (J^{1/2} \tilde{R}_2)(v, \eta) + J^{1/2} \left(a(v) \langle \eta \rangle^{2\gamma}\right) - a(v) \langle \eta \rangle^{2\gamma}.
\]
To obtain (2.22), we use again (2.7) to get
\[(a^\rho)^\text{Wick} = (a^\rho)^w + \tilde{f}^w,\]
where
\[\tilde{f}(v) = \int_0^1 \int_{R^n} (1 - \theta)(a^\rho)^\nu(v + \theta \bar{v})\bar{v}^2 e^{-2\pi i|v|^2} 2^n d\bar{v} d\theta.\]
In view of (2.13), it follows that, with \(\alpha\) arbitrary multi-index,
\[|\partial^n \tilde{f}| \lesssim \int_0^1 \int_{R^n} (v + \theta \bar{v})^{\gamma \rho} |\bar{v}|^2 e^{-2\pi i|v|^2} 2^n d\bar{v} d\theta \lesssim \langle v \rangle^{\gamma \rho},\]
which gives \(\tilde{f}(v, \eta) \in \mathcal{S}(\langle v \rangle^{\gamma \rho}, |dv|^2 + |d\eta|^2)\). On the other hand, by (1.2) and (2.12), we could verify that, with some integer \(k > 0\) depending only on \(\gamma\) and \(s\),
\[a^\rho \in \mathcal{S}(\langle v \rangle^k, |dv|^2 + \langle v \rangle^{-2} |d\eta|^2).\]
As a result, for any positive integer \(M\), applying (2.2) and (2.3) implies
\[(a^\rho)^w = \mathcal{O}_0(J^{1/2}a^\rho) = \mathcal{O}_0(a^\rho + \ell_M) = a^\rho + \mathcal{O}_0(\ell_M),\]
with \(\ell_M \in \mathcal{S}(\langle v \rangle^{-M}, |dv|^2 + \langle v \rangle^{-2} |d\eta|^2)\). Now we may choose \(M\) large enough such that
\[\mathcal{S}(\langle v \rangle^{-M}, |dv|^2 + \langle v \rangle^{-2} |d\eta|^2) \subset \mathcal{S}(\langle v \rangle^{\gamma \rho}, |dv|^2 + \langle v \rangle^{-2} |d\eta|^2),\]
and then let \(r = J^{1/2} + \ell_M\). This gives (2.22). Similarly we can show (2.23). Indeed, using the estimate
\[\forall \ |\alpha| \geq 0, \ |\partial^n \left(N + |v|^2\right)^{\rho/2}| \leq C_\alpha \left(N + |v|^2\right)^{(\rho - |\alpha|)/2},\]
we could repeat the above arguments, with \(a^\rho\) replaced by \(\left(N + |v|^2\right)^{\rho/2}\), to obtain (2.23). The proof is completed. \(\square\)

**Lemma 2.4** Let \(\ell \in \mathcal{S}(\langle v \rangle^{\gamma/2}, |dv|^2 + |d\eta|^2)\) be given, and let \(\widehat{F}\) be the operator defined in (2.1) with \(a, b\) satisfying the assumptions (1.2) and (1.3). Then for any \(f \in \mathcal{S}(R^d)\) we have
\[\|a^{1/2} \langle D_v \rangle f \|_{L^2}^2 + \|\ell(v, D_v) f \|_{L^2}^2 + \|\langle v \rangle^{\gamma/2} f \|_{L^2}^2 \lesssim \text{Re} (\widehat{P} f, f)_{L^2} + \|f\|_{L^2}^2. \quad (2.24)\]

**Proof.** We may write
\[\ell(v, D_v) = (\ell(v, D_v) \langle D_v \rangle^{-1} \langle v \rangle^{-\gamma/2}) \langle v \rangle^{\gamma/2} \langle D_v \rangle^\gamma,\]
with
\[\ell(v, D_v) \langle D_v \rangle^{-1} \langle v \rangle^{-\gamma/2} \in \mathcal{O}_0 \left(S(1, |dv|^2 + |d\eta|^2)\right)\]
due to (2.5). This, along with the \(L^2\) continuity theorem in the class \(S(1, |dv|^2 + |d\eta|^2)\), gives
\[\forall f \in \mathcal{S}(R^d), \quad \|\ell(v, D_v) f \|_{L^2} \lesssim \|\langle v \rangle^{\gamma/2} \langle D_v \rangle f \|_{L^2} \lesssim \|a^{1/2} \langle D_v \rangle f \|_{L^2},\]
the last inequality using (1.2). So we only need to treat the first and third terms on the left-hand side of (2.24). Since the operator \(i (\tau + v \cdot \xi)\) is skew-adjoint, it then follows that
\[\text{Re} (a(v) \langle D_v \rangle^{2\gamma} f, f)_{L^2} + \text{Re} (b(v) f, f)_{L^2} = \text{Re} (\widehat{P} f, f)_{L^2}.\]
Note that \(a, b\) are real-valued functions. Then by virtue of the relation
\[\text{Re} (a(v) \langle D_v \rangle^{2\gamma} f, f)_{L^2} = \text{Re} ([\langle D_v \rangle^\gamma a(v) \langle D_v \rangle^\gamma f, f]_{L^2} - \text{Re} ([\langle D_v \rangle^\gamma, a] \langle D_v \rangle^\gamma f, f)_{L^2},\]
we have
\[\|a^{1/2} \langle D_v \rangle f \|_{L^2}^2 + \|b^{1/2} f \|_{L^2}^2 \lesssim \text{Re} (\widehat{P} f, f)_{L^2} + \text{Re} ([\langle D_v \rangle^\gamma, a] \langle D_v \rangle^\gamma f, f)_{L^2}. \quad (2.25)\]
Moreover, in view of Theorem 2.3.19 in [27] (see Theorem A.1 in the Appendix), we see

\[
[(D_v)^s, a] (D_v)^s = \left( \frac{1}{2i\pi} \text{op}_0(\partial_v a \cdot \partial_n (\eta)^s) + r_1(v, D_v) \right) (D_v)^s
\]

\[= \frac{1}{2i\pi} \text{op}_0((\eta)^s \partial_v a \cdot \partial_n (\eta)^s) + r_1(v, D_v) (D_v)^s,
\]

with

\[r_1(v, \eta) \in S \left( (v)^{2s+\gamma - 2} (\eta)^s \gamma - |v|^2 (\eta)^s \gamma \right) \subset S \left( (v)^{\gamma}, |v|^2 + |\eta|^2 \right).
\]

On the other hand, using (2.2) and (2.3) gives

\[\text{op}_0((\eta)^s \partial_v a \cdot \partial_n (\eta)^s) = (\eta)^s \partial_v a \cdot \partial_n (\eta)^s \| + \varepsilon_{w},
\]

with

\[r_2(v, \eta) = J^{-1/2} (\eta)^s \partial_v a \cdot \partial_n (\eta)^s - (\eta)^s \partial_v a \cdot \partial_n (\eta)^s \in S \left( (v)^{\gamma}, |v|^2 + |\eta|^2 \right).
\]

Thus we may write

\[[(D_v)^s, a] (D_v)^s = \frac{1}{2i\pi} \left( (\eta)^s \partial_v a \cdot \partial_n (\eta)^s \right)^w + \varepsilon_{w},
\]

where the last inequality follows from the interpolation inequality that

\[\forall \ v > 0, \ \left\| (v)^{\gamma} \| f \|_{L^2} \leq \varepsilon \left\| (v)^{s+ \gamma} f \right\|_{L^2} + c \| f \|_{L^2}.
\]

These inequalities along with (1.2) and (2.25) yield

\[\left\| a^2 \langle D_v \rangle^s f \right\|_{L^2} + \left\| (v)^{s+ \gamma} f \right\|_{L^2} \leq \varepsilon \left\| (v)^{s+ \gamma} f \right\|_{L^2} + c \| f \|_{L^2}.
\]

Taking \( \varepsilon \) sufficiently small, we get the desired estimate (2.24), completing the proof.

**Remark 2.5** Let \( \rho(\eta) \) be the function given in (1.4), and denote by \( \tilde{D}_v \) the Fourier multiplier of the symbol \( |\eta|^s \rho(\eta) \). Then repeating the above arguments used in the proof of (2.24), with \( (D_v)^s \) replaced by \( \tilde{D}_v \), we have

\[\left\| a^2 \tilde{D}_v f \right\|_{L^2} + \left\| (v)^{s+ \gamma} f \right\|_{L^2} \leq \varepsilon \left\| (v)^{s+ \gamma} f \right\|_{L^2} + \text{Re} \left( \tilde{P} f, f \right)_{L^2} + c \| f \|_{L^2}.
\]

On the other hand, using the notation \( q(v, D_v) = |\tilde{D}_v|^2 - (-\Delta_v)^s \), we compute

\[\text{Re}(a(v)q(v, D_v) f, f)_{L^2} = \text{Re}(a^{1/2} q(v, D_v) a^{1/2} f, f)_{L^2} - \text{Re}(a^{1/2} [q(v, D_v), a^{1/2}] f, f)_{L^2} \leq -\text{Re}(a^{1/2} [q(v, D_v), a^{1/2}] f, f)_{L^2}.
\]
Moreover, similar to (2.19), we may write
\[ a^{1/2} \left[ q(v, D_v) \right] a^{1/2} = R(v, D_v) + r(v, D_v) \]
with \( R(v, \eta) \in S \left( (v)^{\gamma + \gamma'}, |dv|^2 + |d\eta|^2 \right) \) and \( r \in S \left( (v)^{\gamma/2}, |dv|^2 + |d\eta|^2 \right) \). This implies
\[
-\text{Re} \left[ a^{1/2} \left[ q(v, D_v) \right] f, f \right]_{L^2} \lesssim \left\| (v)^{\gamma + \frac{3}{2}} f \right\|_{L^2} \left\| (v)^{\frac{3}{2}} f \right\|_{L^2} + \left\| f \right\|_{L^2} \left\| (v)^{\frac{3}{2}} f \right\|_{L^2},
\]
and thus, with \( \varepsilon > 0 \) arbitrarily small,
\[
-\text{Re} \left[ a(v) \left( (\tilde{D}_v)^2 - (-\Delta_v)^s \right) f, f \right]_{L^2} \lesssim \varepsilon \left\| (v)^{\gamma + \frac{1}{2}} f \right\|_{L^2}^2 + C_\varepsilon \left\| f \right\|_{L^2}^2.
\]
Combining these inequalities, we conclude
\[
\left\| a^{1/2} |\tilde{D}_v|^s f \right\|_{L^2}^2 + \left\| (v)^{\gamma + \frac{3}{2}} f \right\|_{L^2}^2 \lesssim \text{Re} \left[ f \right]_{L^2} \left( \varepsilon (\tau + v \cdot \xi) + a(v)(-\Delta_v)^s + b(v) \right] f \right\|_{L^2}^2 + \left\| f \right\|_{L^2}^2.
\]
(2.26)

**Corollary 2.6** Let \( \bar{P} \) be the operator given in (2.1) with \( a, b \) satisfying the assumptions (1.2) and (1.3). Suppose \( p \in S \left( (1, |dv|^2 + |d\eta|^2) \right) \) and
\[
R \in S \left( (v)^{\gamma}(\eta)^{2s}, |dv|^2 + |d\eta|^2 \right) \cup S \left( (v)^{\gamma + \gamma'}(\eta)^s, |dv|^2 + |d\eta|^2 \right) \cup S \left( (v)^{2\gamma + \gamma'}, |dv|^2 + |d\eta|^2 \right).
\]
Then we have
\[
\forall f \in S(\mathbb{R}^n), \quad \left\| (a(v) \langle D_v \rangle^{2s} f + R(v, D_v) f, p^{w} f \right\|_{L^2} \lesssim \left\| (\bar{P} f, f \right\|_{L^2} + \left\| f \right\|_{L^2}^2.
\]
(2.27)

**Proof.** Suppose \( R \in S \left( (v)^{\gamma}(\eta)^{2s}, |dv|^2 + |d\eta|^2 \right) \) without loss of generality. Then one may write
\[
p^{w} R(v, D_v) = \langle D_v \rangle^{s} \left( (v)^{\gamma/2} \langle D_v \rangle^{-s} \tilde{p}(v, D_v) R(v, D_v) \langle D_v \rangle^{-s} (v)^{-\gamma/2} \right) (v)^{\gamma/2} \langle D_v \rangle^{s},
\]
where \( \tilde{p} = J^{1/2}p \in S \left( (1, |dv|^2 + |d\eta|^2) \right) \) such that \( p^{w} = \tilde{p}(v, D_v) \). As a result, we have, by the \( L^2 \) continuity in the class \( S \left( (1, |dv|^2 + |d\eta|^2) \right) \),
\[
\left\| (R(v, D_v) f, p^{w} f \right\|_{L^2} \lesssim \left\| (\tilde{p} f, f \right\|_{L^2} + \left\| f \right\|_{L^2}^2.
\]
(2.24) yields
\[
\left\| (R(v, D_v) f, p^{w} f \right\|_{L^2} \lesssim \left\| (\bar{P} f, f \right\|_{L^2} + \left\| f \right\|_{L^2}^2.
\]
Thus, in view of (2.24), the desired estimate (2.27) will follow if we could show that
\[
\left\| (a(v) \langle D_v \rangle^{2s} f, p^{w} f \right\|_{L^2} \lesssim \left\| a^{1/2} \langle D_v \rangle^{s} f \right\|_{L^2}^2 + \left\| (v)^{\frac{3}{2}} \langle D_v \rangle^{s} f \right\|_{L^2}^2 + \left\| (v)^{\gamma + \frac{3}{2}} f \right\|_{L^2}^2.
\]
(2.28)

Observe that the term on the left-hand side is bounded from above by
\[
\left\| (\langle D_v \rangle^{s} a(v) \langle D_v \rangle^{s} f, p^{w} f \right\|_{L^2} + \left\| (\langle D_v \rangle^{s}, a(v)) \langle D_v \rangle^{s} f, p^{w} f \right\|_{L^2},
\]
and moreover
\[
\left\| (\langle D_v \rangle^{s}, a(v)) \langle D_v \rangle^{s} f, p^{w} f \right\|_{L^2} = \left\| (p^{w}[\langle D_v \rangle^{s}, a(v)] \langle D_v \rangle^{s} f, f \right\|_{L^2} \lesssim \left\| (v)^{\gamma/2} \langle D_v \rangle^{s} f \right\|_{L^2} \left\| (v)^{\gamma + \frac{3}{2}} f \right\|_{L^2},
\]
where the last inequality holds because we may write
\[
p^{w}[\langle D_v \rangle^{s}, a(v)] = (v)^{\gamma + \gamma/2} \left( (v)^{-\gamma/2} p^{w}[\langle D_v \rangle^{s}, a(v)] (v)^{\gamma/2} \right) (v)^{\gamma/2}.
\]
(2.26)
Then we have
\[
\left\| (a(v) (D_v)^{2s} f, p^w f)_{L^2} \right\| \lesssim \left\| a^{\frac{1}{2}} (D_v)^{s} f \right\|_{L^2}^2 + \left\| a^{\frac{1}{2}} (D_v)^{s} p^w f \right\|_{L^2}^2 + \left\| (v)^{s+\frac{3}{2}} f \right\|_{L^2}^2.
\]
So in order to obtain (2.28), we only need to treat the second term in the last inequality, which is bounded from above by
\[
\left\| a^{\frac{1}{2}} [ (D_v)^{s}, p^w ] f \right\|_{L^2}^2 + \left\| a^{\frac{1}{2}} (D_v)^{s} f \right\|_{L^2}^2.
\]  
(2.29)

As for the first term in (2.29), we have, by (1.2),
\[
\left\| a^{\frac{1}{2}} [ (D_v)^{s}, p^w ] f \right\|_{L^2}^2 \lesssim \left\| (v)^{s+\frac{3}{2}} [ (D_v)^{s}, p^w ] f \right\|_{L^2}^2 \lesssim \left\| (v)^{s+\frac{3}{2}} f \right\|_{L^2}^2,
\]
where the last inequality holds because applying Theorem 2.3.8 in [27] (see Theorem A.4 in the Appendix), we have
\[
[ (D_v)^{s}, p^w ] \in \text{op}_{1/2} \left( S \left( 1, |dv|^2 + |d\eta|^2 \right) \right),
\]
and thus
\[
(v)^{s+\frac{3}{2}} [ (D_v)^{s}, p^w ] = \left( v^{s+\frac{3}{2}} \left[ (D_v)^{s}, p^w \right] v^{-s-\frac{3}{2}} \right) (v)^{s+\frac{3}{2}}.
\]
As for the second term in (2.29), by (2.20) we have
\[
\left\| a^{\frac{1}{2}} (D_v)^{s} f \right\|_{L^2}^2 \lesssim \left\| a^{1/2} (D_v)^{s} f \right\|_{L^2}^2 + \left\| (v)^{s/2} (D_v)^{s} f \right\|_{L^2}^2.
\]
Combining the above inequalities, we get (2.28), completing the proof.  

\( \square \)

**Corollary 2.7** Let \( \hat{\mathcal{P}} \) be the operator given in (2.1) with \( a, b \) satisfying the assumptions (1.2) and (1.3). Then
\[
\forall f \in \mathcal{S}(\mathbb{R}^n), \quad \left\| \left( (a(v) (\eta)^2) \text{Wick } f, f \right)_{L^2} \right\| \lesssim \left\| (\hat{\mathcal{P}} f, f)_{L^2} \right\| + \left\| f \right\|_{L^2}^2.
\]  
(2.30)

**P r o o f.** This is just a consequence of (2.21) and (2.27).  

\( \square \)

## 3 Proof of the main results

In this section we will prove Theorem 1.1. As mentioned at the beginning of Subsection 2.1, it suffices to study the operator \( \hat{\mathcal{P}} \) defined in (2.1), and Theorem 1.1 will follow immediately from the following result concerning the hypoellipticity of the operator with parameters. Since our main analysis is still on \( \mathbb{R}^n \), we will consider \( \xi \) and \( \tau \) in the operator \( \hat{\mathcal{P}} \) as parameters and use the same notation as in the previous section; that is, \( (\cdot, \cdot)_{L^2} \) and \( \| \cdot \|_{L^2} \) stand for \( (\cdot, \cdot)_{L^2(\mathbb{R}^n)} \) and \( \| \cdot \|_{L^2(\mathbb{R}^n)} \), respectively.

**Theorem 3.1** Let \( \hat{\mathcal{P}} \) be the operator given in (2.1) with \( a, b \) satisfying the assumptions (1.2) and (1.3). Then for all \( f \in \mathcal{S}(\mathbb{R}^n) \) we have
\[
\left\| (v)^{s_1} a^{\frac{s_1}{2}} \tau^{s_1} \left( \tau \right)^{\frac{s_1}{2}} f \right\|_{L^2} + \left\| a^{\frac{s_1}{2}} \left( \eta \right)^{\frac{s_1}{2}} f \right\|_{L^2} + \left\| a (v) (D_v)^{2s_1} f \right\|_{L^2} + \left\| (v)^{s_2 + \frac{s_1}{2}} f \right\|_{L^2} \lesssim \left\| \hat{\mathcal{P}} f \right\|_{L^2} + \left\| f \right\|_{L^2},
\]
which holds uniformly with respect to \( \xi \) and \( \tau \).
3.1 The first part of the proof of Theorem 3.1

In this subsection we prove the following

**Proposition 3.2** Let $\omega_1 \in S((v)^{s+\gamma} |dv|^s + |d\eta|^2)$ and $\omega_2 \in S((v)^{2s+\gamma} |dv|^2 + |d\eta|^2)$ be given, and let $\tilde{\mathcal{P}}$ be the operator defined in (2.1) with $a$, $b$ satisfying the assumptions (1.2) and (1.3). Then the following estimate

$$\forall f \in S(\mathbb{R}^n), \quad \|\omega_1(v, D_v) f \|_{L^2} + \|\omega_2(v, D_v) f \|_{L^2} \lesssim \|\tilde{\mathcal{P}} f \|_{L^2} + \| f \|_{L^2}$$

(3.1)

holds uniformly with respect to $\xi$ and $\tau$.

**Proof.** Suppose $f \in S(\mathbb{R}^n)$. Let’s firstly prove that, with $\varepsilon > 0$ arbitrarily small,

$$\text{Re} \left( (v)^{s+\gamma} \left[ \tilde{\mathcal{P}}, \langle v \rangle^{s+\gamma} f, f \right] \right)_{L^2} \lesssim \varepsilon \| (v)^{2s+\gamma} f \|_{L^2}^2 + C_\varepsilon \| f \|_{L^2}^2.$$  

(3.2)

Indeed, in view of Theorem 2.3.19 in [27] (see Theorem A.1 in the Appendix), we see

$$\langle v \rangle^{s+\gamma} \left[ \tilde{\mathcal{P}}, \langle v \rangle^{s+\gamma} f, f \right] = \langle v \rangle^{s+\gamma} a(v) \left[ (D_v)^{2s} \langle v \rangle^{s+\gamma} f, f \right] + r_1(v, D_v),$$

$$= \langle v \rangle^{s+\gamma} a(v) \left( \frac{1}{2L^\pi} \text{op}_0 \left( \langle \eta \rangle^{s+\gamma} \partial_\eta \cdot \partial_\eta \langle v \rangle^{s+\gamma} f, f \right) \right) + \langle v \rangle^{s+\gamma} a(v) r_1(v, D_v),$$

with

$$r_1(v, \eta) \in S((v)^{s+\gamma/2-2} \langle v \rangle^{2s-2} + \langle v \rangle^{-2} |dv|^2 + \langle \eta \rangle^{-2} |d\eta|^2) \subset S((v)^{s+\gamma}, |dv|^2 + |d\eta|^2).$$

On the other hand, using (2.2) gives

$$\text{op}_0 \left( \langle v \rangle^{s+\gamma} a(v) \partial_\eta \langle v \rangle^{s+\gamma} \partial_\eta \langle v \rangle^{s+\gamma} f, f \right) = \left( \langle v \rangle^{s+\gamma} a(v) \partial_\eta \langle v \rangle^{s+\gamma} \partial_\eta \langle v \rangle^{s+\gamma} f, f \right) + r_2(v, \eta),$$

with

$$r_2(v, \eta) = (v)^{s+\gamma} \left[ \partial_\eta \langle v \rangle^{s+\gamma} f, f \right] - \langle v \rangle^{s+\gamma} a(v) \partial_\eta \langle v \rangle^{s+\gamma} \partial_\eta \langle v \rangle^{s+\gamma} f,$$

which belongs to $S((v)^{3s+2\gamma}, (v)^{-2} |dv|^2 + \langle \eta \rangle^{-2} |d\eta|^2)$ due to (2.3). Thus we may write

$$\langle v \rangle^{s+\gamma} \left[ \tilde{\mathcal{P}}, \langle v \rangle^{s+\gamma} f, f \right] = \frac{1}{2L^\pi} \left( \langle v \rangle^{s+\gamma} a(v) \partial_\eta \langle v \rangle^{s+\gamma} \partial_\eta \langle v \rangle^{s+\gamma} f, f \right) + r_2^w + \langle v \rangle^{s+\gamma} a(v) r_1(v, D_v).$$

This implies, observing the operator $\frac{1}{2L^\pi} \left( \langle v \rangle^{s+\gamma} a(v) \partial_\eta \langle v \rangle^{s+\gamma} \partial_\eta \langle v \rangle^{s+\gamma} f, f \right)$ is skew-adjoint,

$$\text{Re} \left( (v)^{s+\gamma} \left[ \tilde{\mathcal{P}}, \langle v \rangle^{s+\gamma} f, f \right] \right)_{L^2} = \text{Re} \left( r_2^w f + \langle v \rangle^{s+\gamma} a(v) r_1(v, D_v) f, f \right)_{L^2} \lesssim \| (v)^{2s+\gamma} f \|_{L^2}^2 + \langle v \rangle^{s+\gamma} f \|_{L^2},$$

where the last inequality holds because we may write

$$\langle v \rangle^{s+\gamma} a(v) r_1(v, D_v) = \langle v \rangle^{2s+\gamma} \langle v \rangle^{-(2s+\gamma)} \langle v \rangle^{s+\gamma} a(v) r_1(v, D_v) \langle v \rangle^{-(s+\gamma)} \langle v \rangle^{s+\gamma}$$

and

$$r_2^w = \langle v \rangle^{2s+\gamma} \langle v \rangle^{-(2s+\gamma)} r_2^w \langle v \rangle^{-(s+\gamma)} \langle v \rangle^{s+\gamma},$$

$$\in \text{op}_0 \left( s(1, |dv|^2 + |d\eta|^2) \right)$$

On the other hand, the interpolation inequality gives

$$\| (v)^{s+\gamma} f \|_{L^2} \leq \varepsilon \| (v)^{2s+\gamma} f \|_{L^2}^2 + C_\varepsilon \| f \|_{L^2}.$$
due to the fact that $s > 0$. Combining these inequalities, we obtain (3.2).

Observe that
\[
\| (v)^{s+\gamma} (D_v)^s \|_{L^2}^2 \lesssim \| (v)^{s+\gamma/2} (D_v)^s (v)^{s+\gamma/2} f \|_{L^2}^2 + \| (v)^{s+\gamma/2} (D_v)^s f \|_{L^2}^2 \lesssim \| (v)^{s+\gamma/2} (D_v)^s f \|_{L^2}^2 + \| (v)^{\gamma/2} f \|_{L^2}^2,
\]
the last inequality following from (2.10). Then we use (2.24) to the function $(v)^{s+\gamma/2} f$, to obtain
\[
\| (v)^{s+\gamma} (D_v)^s f \|_{L^2}^2 + \| (v)^{2s+\gamma} f \|_{L^2}^2 \\
\lesssim \text{Re} \left( \tilde{\mathcal{P}} (v)^{s+\gamma/2} f, (v)^{s+\gamma/2} f \right)_{L^2} + \| (v)^{\gamma/2} f \|_{L^2}^2 \\
\lesssim \text{Re} \left( \tilde{\mathcal{P}} f, (v)^{2s+\gamma} f \right)_{L^2} + \text{Re} \left( (v)^{s+\gamma/2} \left[ \tilde{\mathcal{P}}, (v)^{s+\gamma/2} \right] f, f \right)_{L^2} + \| (v)^{\gamma/2} f \|_{L^2}^2,
\]
which, along with the interpolation inequality, implies that
\[
\| (v)^{s+\gamma} (D_v)^s f \|_{L^2} + \| (v)^{2s+\gamma} f \|_{L^2} \lesssim \| \tilde{\mathcal{P}} f \|_{L^2}^2 + \text{Re} \left( (v)^{s+\gamma/2} \left[ \tilde{\mathcal{P}}, (v)^{s+\gamma/2} \right] f, f \right)_{L^2}.
\]
Thus by (3.2) we have
\[
\| (v)^{s+\gamma} (D_v)^s f \|_{L^2} + \| (v)^{2s+\gamma} f \|_{L^2} \lesssim \| \tilde{\mathcal{P}} f \|_{L^2}^2 + \| f \|_{L^2}^2.
\]
Using again the $L^2$ continuity in the class $S(1, g)$, we get the desired estimate (3.1), since we may write
\[
\omega_1(v, D_v) = \left( \omega(v, D_v) (D_v)^{-s} (v)^{-s} \right) (v)^{s+\gamma} (D_v)^s \\
\quad \quad \in \mathcal{O}(1, |\partial v|^2 + |\partial \eta|^2)
\]
and
\[
\omega_2(v, D_v) = \left( \omega(v, D_v) (v)^{-2s+\gamma} \right) (v)^{2s+\gamma}. \\
\quad \quad \in \mathcal{O}(1, |\partial v|^2 + |\partial \eta|^2)
\]
The proof is complete. \hfill \Box

\textbf{Remark 3.3} The estimate (2.26) allows us to repeat the above arguments to get
\[
\| (v)^{s+\gamma} (D_v)^s f \|_{L^2} + \| (v)^{2s+\gamma} f \|_{L^2} \lesssim \left\| \tilde{\mathcal{I}} (\tau + v \cdot \xi) + a(v) \left( - \tilde{\Delta}_v \right)^s + b(v) \right\|_{L^2} + \| f \|_{L^2}.
\]

\subsection{The second part of the proof of Theorem 3.1}
In this subsection we make use of the multiplier method used in [22], [29] to prove the following

\textbf{Proposition 3.4} Let $\tilde{\mathcal{P}}$ be given in (2.1) with $a$, $b$ satisfying the assumptions (1.2) and (1.3). Then the following estimate
\[
\forall f \in \mathcal{S}(\mathbb{R}^n), \quad \{ a \frac{\tau \cdot \xi}{|\xi|^2} \tilde{\mathcal{P}} f \} \lesssim \| \tilde{\mathcal{P}} f \|_{L^2} + \| f \|_{L^2} 
\]
holds uniformly with respect to $\xi$ and $\tau$.

\textbf{Proof.} Suppose $f \in \mathcal{S}(\mathbb{R}^n)$. We proceed to prove Proposition 3.4 through four steps.

\textit{Step 1:} In what follows let $\xi \in \mathbb{R}^n$ be fixed, and we define a symbol $p$ by setting
\[
p(v, \eta) = p_\xi(v, \eta) = \frac{a \frac{\tau \cdot \xi}{|\xi|^2} \xi \cdot \eta}{{|\xi|^2}} \psi.
\]


with \( a, s \) given in (1.1) and \( \psi \) defined by
\[
\psi(v, \eta) = \chi\left(\frac{a(v) \langle\eta\rangle^{1+2s}}{\langle\xi\rangle}\right),
\]
where \( \chi \in C_0^\infty(\mathbb{R}; [0, 1]) \) such that \( \chi = 1 \) in \([-1, 1] \) and \( \text{supp} \, \chi \subset [-2, 2] \). Then by virtue of (2.12) we can verify that
\[
p, \, \psi \in S\{1, |dv|^2 + |d\eta|^2\}
\]
uniformly with respect to \( \xi \). Moreover we have
\[
|\xi \cdot \partial_\eta \psi| \leq (1 + 2s)\|\chi\|_{L^\infty} \frac{a(v) \langle\eta\rangle^{2s-1} |\xi \cdot \eta|}{\langle\xi\rangle} \lesssim a(v) \langle\eta\rangle^{2s}.
\]

**Step 2:** Let \( p^{\text{Wick}} \) be the Wick quantization of the symbol \( p \) given in (3.4). Then by (2.7) and (3.5) we can find a symbol \( \tilde{p}^w = p^{\text{Wick}} \) such that \( \tilde{p}^w = p \). In this step we will prove
\[
\left\| a \frac{i}{\pi \tau} \langle\xi\rangle \frac{\psi}{\tau} \left\|_{L^2}^2 \lesssim \left(\tilde{p} f, f\right)_{L^2} + \left( a(v) \langle D_\xi\rangle^{2s} f + b f, \, p^{\text{Wick}} f \right)_{L^2} \right. \right.
\]
To do so we make use of the relations
\[
\text{Re} \left( i (\tau + v \cdot \xi) f, \, p^{\text{Wick}} f \right)_{L^2} = \text{Re} \left( \tilde{p} f, \, p^{\text{Wick}} f \right)_{L^2} - \text{Re} \left( a(v) \langle D_\xi\rangle^{2s} f + b f, \, p^{\text{Wick}} f \right)_{L^2}
\]
and
\[
\left| a(v) \langle D_\xi\rangle^{2s} f + b f, \, p^{\text{Wick}} f \right|_{L^2} \lesssim \left| \left( a(v) \langle D_\xi\rangle^{2s} f + b f, \, \tilde{p}^w f \right)_{L^2} \right|_{L^2} \]
due to (2.27), to conclude
\[
\text{Re} \left( i (\tau + v \cdot \xi) f, \, p^{\text{Wick}} f \right)_{L^2} \lesssim \left(\tilde{p} f, \, f\right)_{L^2} + \left(\tilde{p} f, \, p^{\text{Wick}} f \right)_{L^2} + \left| f \right|_{L^2}^2
\]
Next we will give a lower bound of the term on the left-hand side of (3.8). Observe that by (2.7),
\[
\tau + v \cdot \xi = (\tau + v \cdot \xi)^{\text{Wick}}
\]
Then we have, by (2.9),
\[
\text{Re} \left( i (\tau + v \cdot \xi) f, \, p^{\text{Wick}} f \right)_{L^2} = \frac{1}{4\pi} \left(\{ p, \, \tau + v \cdot \xi \}^{\text{Wick}} f, \, f \right)_{L^2}
\]
where \( \{, \, \} \) is the Poisson bracket defined in (2.8). Direct calculus shows
\[
\left\{ p, \, \tau + v \cdot \xi \right\} = a \frac{i}{\tau} \left[ \frac{\xi}{(\xi)^2 - \frac{\tau^2}{\tau^2}} \frac{\psi}{\tau} \right] + a \frac{i}{\tau} \left[ \frac{\xi \cdot \eta}{(\xi)^2 - \frac{\tau^2}{\tau^2}} \right] \langle\xi\rangle^2 \frac{\psi}{\tau}
\]
\[
= \frac{i}{\tau} \langle\xi\rangle \frac{\psi}{\tau} - a \frac{\xi}{(\xi)^2 - \frac{\tau^2}{\tau^2}} \langle\xi\rangle^2 \frac{\psi}{\tau}
\]
\[
= \frac{i}{\tau} \langle\xi\rangle \frac{\psi}{\tau} - a \frac{\xi}{(\xi)^2 - \frac{\tau^2}{\tau^2}} \langle\xi\rangle^2 \frac{\psi}{\tau} \left(1 \cdot \right)
\]
\[
= - \frac{i}{\tau} \langle\xi\rangle \frac{\psi}{\tau} + a \frac{\xi}{(\xi)^2 - \frac{\tau^2}{\tau^2}} \langle\xi\rangle^2 \frac{\psi}{\tau} \cdot \partial_\psi.
\]
The above equalities along with (3.8) and (3.9) yield
\[
\left( a \frac{i}{\tau} \langle\xi\rangle \frac{\psi}{\tau} \left)_{L^2} \lesssim \sum_{j=1}^3 K_j \right. \left|\tilde{p} f, \, f\right|_{L^2} + \left|\tilde{p} f, \, p^{\text{Wick}} f \right|_{L^2} + \left| f \right|_{L^2}^2
\]
with

\[ K_1 = \left( a \frac{\psi}{\langle \xi \rangle} \langle \xi \rangle^{2s/(1 + 2s)} (1 - \psi) \right)_{\text{Wick}} f, f \right)_{L^2}, \]

\[ K_2 = \left( a \frac{\psi}{\langle \xi \rangle} \langle \xi \rangle^{2s/(1 + 2s)} f, f \right)_{L^2}, \]

\[ K_3 = \left( a \frac{\xi \cdot \eta}{\langle \xi \rangle} \langle \xi \rangle^{2s/(1 + 2s)} f, f \right)_{L^2}. \]

Note that \( \langle \xi \rangle^{2s/(1 + 2s)} \leq a^{2s/(1 + 2s)} \langle \eta \rangle^{2s} \) on the support of \( 1 - \psi \), and thus
\[ a \frac{\psi}{\langle \xi \rangle} \langle \xi \rangle^{2s/(1 + 2s)} (1 - \psi) \leq a(v) \langle \eta \rangle^{2s}. \]

Then the positivity of Wick quantization gives
\[ K_1 \lesssim \left( (a(v) \langle \eta \rangle^{2s})_{\text{Wick}} f, f \right)_{L^2} \lesssim \left| (\bar{\mathcal{P}} f, f)_{L^2} \right| + \left\| f \right\|_{L^2}^2, \] (3.11)
where the last inequality follows from (2.30). Similarly, observing
\[ a \frac{\psi}{\langle \xi \rangle} \langle \xi \rangle^{1/(1 + 2s)} \lesssim \langle \xi \rangle^{2s/(1 + 2s)} \leq 1, \]
and
\[ -a \frac{\xi \cdot \eta}{\langle \xi \rangle} \langle \xi \rangle^{2s/(1 + 2s)} \lesssim a(v) \langle \eta \rangle^{2s} \]
due to (3.6), we have
\[ K_2 + K_3 \lesssim \left( (a(v) \langle \eta \rangle^{2s})_{\text{Wick}} f, f \right)_{L^2} + \left\| f \right\|_{L^2}^2 \lesssim \left| (\bar{\mathcal{P}} f, f)_{L^2} \right| + \left\| f \right\|_{L^2}^2. \]
This, together with (3.10) and (3.11), gives
\[ \left( a \frac{\psi}{\langle \xi \rangle} \langle \xi \rangle^{2s/(1 + 2s)} f, f \right)_{L^2} \lesssim \left| (\bar{\mathcal{P}} f, f)_{L^2} \right| + \left| (\bar{\mathcal{P}} f, p_{\text{Wick}} f)_{L^2} \right| + \left\| f \right\|_{L^2}^2. \] (3.12)
On the other hand, it follows from (2.22) that
\[ \left( a \frac{\psi}{\langle \xi \rangle} \langle \xi \rangle^{2s/(1 + 2s)} f, f \right)_{L^2} = \left( a \frac{\psi}{\langle \xi \rangle} \langle \xi \rangle^{2s/(1 + 2s)} f, f \right)_{L^2} + \left( r(v, D_v) \langle \xi \rangle^{2s/(1 + 2s)} f, f \right)_{L^2}, \]
where \( r \in S \left( v \langle \eta \rangle^{2s}, |dv|^2 + |d\eta|^2 \right) \). As a result, writing
\[ r(v, D_v) = \langle v \rangle^{2s} \left( (v)^{-2s} r(v, D_v) (v)^{-2s} \right) (v), \]
and then using the \( L^2 \) continuity theorem in the class \( S \left( 1, |dv|^2 + |d\eta|^2 \right) \), we obtain
\[ \left( a \frac{\psi}{\langle \xi \rangle} \langle \xi \rangle^{2s/(1 + 2s)} f, f \right)_{L^2} \lesssim \left( a \frac{\psi}{\langle \xi \rangle} \langle \xi \rangle^{2s/(1 + 2s)} f, f \right)_{L^2} + \left\| (v) \langle \eta \rangle^{2s} \langle \xi \rangle^{2s} f \right\|_{L^2}^2, \]
which together with (3.12) gives the desired estimate (3.7).

Step 3: Next we show that there exists a symbol \( q \in S \left( 1, |dv|^2 + |d\eta|^2 \right) \) such that
\[ \left\| (v) \langle \eta \rangle^{2s} \langle \xi \rangle^{2s} f \right\|_{L^2}^2 \lesssim \left| (\bar{\mathcal{P}} f, f)_{L^2} \right| + \left| (\bar{\mathcal{P}} f, q_{\text{Wick}} f)_{L^2} \right| + \left\| f \right\|_{L^2}^2. \] (3.13)
For this purpose, let $N$ be a positive integer which is to be determined later, and define

$$ q_N(v, \eta) = q_{\xi, N}(v, \eta) = \frac{(N + |v|^2)^{\gamma/2}}{(\xi)^{N+|\gamma|+2s}} \phi_N, $$

with $\gamma, s$ given in (1.2) and (1.1), and $\phi_N$ defined by

$$ \phi_N(v, \eta) = \chi \left( \frac{(N + |v|^2)^{\gamma/2}}{(\xi)^{N+|\gamma|+2s}} \right). $$

Recall $\chi \in C^\infty_0(\mathbb{R}; [0, 1])$ such that $\chi = 1$ in $[-1, 1]$ and $\text{supp} \chi \subset [-2, 2]$. Direct verification gives

$$ q_N, \quad \phi_N \in S(1, |dv|^2 + |d\eta|^2) $$

(3.14)

uniformly with respect to $\xi$ and $N$. Note that $q_N$ is constructed similarly as $p$ above, with the factor $a(v)$ in (3.4) replaced by $(N + |v|^2)^{\gamma/2}$. Then repeating the arguments used to prove (3.12), we have

$$ \left( (N + |v|^2)^{\gamma/2} \frac{\partial}{\partial v} f \right)^\dagger \lesssim \left( (N + |v|^2)^{\gamma/2} \frac{\partial}{\partial \eta} f \right)^\dagger + \left( (N + |v|^2)^{\gamma/2} \frac{\partial}{\partial \eta} f \right)^\dagger. $$

This along with (2.23) yields

$$ \left( (N + |v|^2)^{\gamma/2} \frac{\partial}{\partial v} f \right)^\dagger \lesssim \left( (N + |v|^2)^{\gamma/2} \frac{\partial}{\partial \eta} f \right)^\dagger + \left( (N + |v|^2)^{\gamma/2} \frac{\partial}{\partial \eta} f \right)^\dagger + \left( (N + |v|^2)^{\gamma/2} \frac{\partial}{\partial \eta} f \right)^\dagger. $$

which allows us to choose an positive integer $N_0$ sufficiently large, such that

$$ \left\| (N_0 + |v|^2)^{\gamma/2} \frac{\partial}{\partial v} f \right\|^2 \lesssim \left\| (\tilde{p} f, f) \right\|_{L^2} + \left\| \tilde{p} f, f \right\|_{L^2} + \left\| (N + |v|^2)^{\gamma/2} \frac{\partial}{\partial \eta} f \right\|^2. $$

Hence the desired estimate (3.13) follows if we choose $q = q_{N_0}$, since

$$ (v)^{\gamma/2} \lesssim C_{N_0, \gamma} (N_0 + |v|^2)^{\gamma/2} $$

with $C_{N_0, \gamma}$ a constant depending only on $N_0, \gamma$.

**Step 4:** Let $\tilde{p}, \tilde{q} \in S(1, |dv|^2 + |d\eta|^2)$ such that $\tilde{p}^w = p^{Wick}$ and $\tilde{q}^w = q^{Wick}$ with $p, q$ given in the previous two steps. Then combining (3.7) and (3.13), we conclude

$$ \left\| a^{\gamma/2} (\xi)^{\gamma/2} f \right\|^2 \lesssim \left\| (\tilde{p} f, f) \right\|_{L^2} + \left\| \tilde{p} f, \omega(v, D_v) f \right\|_{L^2}, $$

where

$$ \omega = J^{1/2} \tilde{p} + J^{1/2} \tilde{q} \in S(1, |dv|^2 + |d\eta|^2) $$

uniformly with respect to $\xi$. Now applying the above inequality to the function $a^{\gamma/2} f$, we get

$$ \left\| a^{\gamma/2} (\xi)^{\gamma/2} f \right\|^2 \lesssim \left\| (\tilde{p} a^{\gamma/2} f, a^{\gamma/2} f) \right\|_{L^2} + \left\| (\tilde{p} a^{\gamma/2} f, \omega(v, D_v) a^{\gamma/2} f) \right\|_{L^2} + \left\| a^{\gamma/2} f \right\|^2 \lesssim \left\| (\tilde{p} a^{\gamma/2} f, a^{\gamma/2} f) \right\|_{L^2} + \left\| (\tilde{p} a^{\gamma/2} f, \omega(v, D_v) a^{\gamma/2} f) \right\|_{L^2} + \left\| a^{\gamma/2} f \right\|^2 \lesssim \left\| (\tilde{p} a^{\gamma/2} f, a^{\gamma/2} f) \right\|_{L^2} + \left\| (\tilde{p} a^{\gamma/2} f, \omega(v, D_v) a^{\gamma/2} f) \right\|_{L^2} + \left\| a^{\gamma/2} f \right\|^2. $$

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\[
\|\tilde{\mathcal{P}} f\|_{L^1} \lesssim \|\tilde{\mathcal{P}} f\|_{L^2} \left| a^{\frac{\tau}{1+\tau}} f \right|_{L^2} + \|\tilde{\mathcal{P}} f\|_{L^2} \left| a^{\frac{\tau}{1+\tau}} \omega(v, D_v) a^{\frac{\tau}{1+\tau}} f \right|_{L^2} + \|a^{\frac{\tau}{1+\tau}} f\|_{L^2} \|f\|_{L^2} \\
+ \left|\left(\tilde{\mathcal{P}}, a^{\frac{\tau}{1+\tau}} f\right)\right|_{L^2} + \left|\left(\tilde{\mathcal{P}}, a^{\frac{\tau}{1+\tau}} f, \omega(v, D_v) a^{\frac{\tau}{1+\tau}} f\right)\right|_{L^2}.
\]

In order to handle the last two terms in the above inequality, we make use of the relation (2.19); this gives, with \(R \in S((v)^{\gamma+\gamma}(|\eta|^2), |dv|^2 + |d\eta|^2)\) and \(r \in S((v)^{\gamma/(2+4\xi)}, |dv|^2 + |d\eta|^2)\),

\[
\left|\left(\tilde{\mathcal{P}}, a^{\frac{\tau}{1+\tau}} f, \omega(v, D_v) a^{\frac{\tau}{1+\tau}} f\right)\right|_{L^2} \lesssim \left|\left(\tilde{\mathcal{P}} f, a^{\frac{\tau}{1+\tau}} f\right)\right|_{L^2} + \left|\left(\tilde{\mathcal{P}} f, a^{\frac{\tau}{1+\tau}} f, \omega(v, D_v) a^{\frac{\tau}{1+\tau}} f\right)\right|_{L^2} + \left|\left(r f, a^{\frac{\tau}{1+\tau}} f\right)\right|_{L^2} + \left|\left(r f, a^{\frac{\tau}{1+\tau}} f, \omega(v, D_v) a^{\frac{\tau}{1+\tau}} f\right)\right|_{L^2}.
\]

where the last inequality follows from (3.1) and (1.2). Similarly,

\[
\left|\left(\tilde{\mathcal{P}}, a^{\frac{\tau}{1+\tau}} f\right)\right|_{L^2} \lesssim \left|\left(\tilde{\mathcal{P}} f\right)\right|_{L^2} + \left|\left(\tilde{\mathcal{P}} f, a^{\frac{\tau}{1+\tau}} f\right)\right|_{L^2} \left| \omega(v, D_v) a^{\frac{\tau}{1+\tau}} f \right|_{L^2}.
\]

Combining these inequalities, we have

\[
\left| a^{\frac{\tau}{1+\tau}} \langle \xi \rangle^{\frac{\tau}{1+\tau}} f \right|_{L^2} \lesssim \left(\|\tilde{\mathcal{P}} f\|_{L^2} + \|f\|_{L^2}\right) \left(\|a^{\frac{\tau}{1+\tau}} f\|_{L^2} + \|a^{\frac{\tau}{1+\tau}} \omega(v, D_v) a^{\frac{\tau}{1+\tau}} f\|_{L^2}\right).
\]

Moreover by virtue of (2.20), it follows that

\[
\left| a^{\frac{\tau}{1+\tau}} \omega(v, D_v) a^{\frac{\tau}{1+\tau}} f \right|_{L^2} \lesssim \left(\|\omega(v, D_v) a^{\frac{\tau}{1+\tau}} f\|_{L^2} + \left| \omega(v, D_v) a^{\frac{\tau}{1+\tau}} f \right|_{L^2}\right)
\]

\[
\lesssim \left| a^{\frac{\tau}{1+\tau}} f \right|_{L^2} + \left| a^{\frac{\tau}{1+\tau}} a^{\frac{\tau}{1+\tau}} f \right|_{L^2} + \|v^{\frac{\tau}{1+\tau}} a^{\frac{\tau}{1+\tau}} f\|_{L^2}
\]

\[
\lesssim \left| a^{\frac{\tau}{1+\tau}} f \right|_{L^2},
\]

the last inequality using (1.2). Thus

\[
\left| a^{\frac{\tau}{1+\tau}} \langle \xi \rangle^{\frac{\tau}{1+\tau}} f \right|_{L^2} \lesssim \left| a^{\frac{\tau}{1+\tau}} f \right|_{L^2} \left(\|\tilde{\mathcal{P}} f\|_{L^2} + \|f\|_{L^2}\right).
\]

Recall that \(\|\cdot\|_{L^2}\) stands for the norm in \(L^2(\mathbb{R}^n)\). Then multiplying both sides the factor \(\langle \xi \rangle^{2\tau/(1+2\tau)}\), we get

\[
\left| a^{\frac{\tau}{1+\tau}} \langle \xi \rangle^{\frac{\tau}{1+\tau}} f \right|_{L^2} \lesssim \left| a^{\frac{\tau}{1+\tau}} \langle \xi \rangle^{\frac{\tau}{1+\tau}} f \right|_{L^2} \left(\|\tilde{\mathcal{P}} f\|_{L^2} + \|f\|_{L^2}\right),
\]

which gives the desired estimate (3.3), completing the proof of Proposition 3.4.

\[\square\]

### 3.3 Completion of the proof of Theorem 3.1

In view of (3.1) and (3.3), the proof of Theorem 3.1 will be completed if we could show the following

**Proposition 3.5** Let \(\tilde{\mathcal{P}}\) be the operator given in (2.1) with \(a, b\) satisfying the assumptions (1.2) and (1.3). Then the following estimate

\[
\forall f \in S(\mathbb{R}^n), \quad \left| (v)^{-\frac{\tau}{2\tau}} a^{\frac{\tau}{1+\tau}} (\tau)^{\frac{\tau}{1+\tau}} f \right|_{L^2} + \|a(v) D_v^{2\tau} f\|_{L^2} \lesssim \|\tilde{\mathcal{P}} f\|_{L^2} + \|f\|_{L^2} \quad (3.15)
\]

holds uniformly with respect to \(\xi\) and \(\tau\).
Proof. Suppose $f \in S(\mathbb{R}^n)$. The proof is divided into five steps.

Step 1: In what follows let $\varepsilon > 0$ be an arbitrarily small number, which is to be determined later, and denote by $C_\varepsilon$ the different suitable constants depending only on $\varepsilon$. We define $\varphi_\varepsilon$ by

$$\varphi_\varepsilon(v, \eta) = \chi \left( \frac{\langle \xi \rangle}{\varepsilon a(v) \langle \eta \rangle^{1+2\varepsilon}} \right),$$

where $\chi \in C_0^\infty(\mathbb{R}; [0, 1])$ such that $\chi = 1$ in $[-1, 1]$ and supp $\chi \subset [-2, 2]$. Let $\lambda_{1, \varepsilon}$ and $\lambda_{2, \varepsilon}$ be two symbols defined by

$$\lambda_{1, \varepsilon}(v, \eta) = \varphi_\varepsilon(v, \eta) \langle \eta \rangle^{2\varepsilon},$$

and

$$\lambda_{2, \varepsilon}(v, \eta) = (1 - \varphi_\varepsilon(v, \eta)) \langle \eta \rangle^{2\varepsilon}.$$  

By (2.12), we can verify that $\varphi_\varepsilon(v, \eta) \in S \{1, |dv|^2 + \langle \eta \rangle^{-2} |d\eta|^2 \}$ uniformly with respect to $\xi$ and $\varepsilon$, and thus

$$\lambda_{1, \varepsilon}, \lambda_{2, \varepsilon} \in S \langle \langle \eta \rangle^{2\varepsilon}, |dv|^2 + \langle \eta \rangle^{-2} |d\eta|^2 \rangle$$

uniformly with respect to $\xi$ and $\varepsilon$. Moreover we have

$$\forall |\alpha| + |\beta| \geq 0, \quad |\partial^\alpha \partial^\beta \lambda_{2, \varepsilon}(v, \eta)| \lesssim \varepsilon^{-\frac{\alpha + \beta}{2\varepsilon}} \langle \xi \rangle^{2\varepsilon} a^{-\frac{\alpha}{1+2\varepsilon}}(v) \langle \eta \rangle^\frac{\alpha}{1+2\varepsilon},$$

which holds uniformly with respect to $\xi$ and $\varepsilon$, since $\langle \eta \rangle^{2\varepsilon} \leq e^{-\frac{\alpha}{1+2\varepsilon}} a^{-\frac{\alpha}{1+2\varepsilon}}(v) \langle \xi \rangle^\frac{\alpha}{1+2\varepsilon}$ on the support of $\lambda_{2, \varepsilon}$.

Step 2: Let $\lambda_{1, \varepsilon}(v, \eta)$ be given in (3.16). In this step we show

$$\left| (a(v)[\tau + v \cdot \xi, \lambda_{1, \varepsilon}(v, D_v)] f, f)_{L^2} \right| \lesssim \varepsilon \left\| (D_v)^{2\varepsilon} a(v) f \right\|_{L^2}^2 + C_\varepsilon \left( \left\| \hat{\Phi} f \right\|_{L^2}^2 + \left\| f \right\|_{L^2}^2 \right).$$

Note that the term on the left-hand side of the above inequality is bounded from above by

$$\left| (a(v)a^{-1}(v)\tilde{\lambda}_{1, \varepsilon}(v, D_v)a(v)f, f)_{L^2} \right| + \left| (a(v)a^{-1}(v)\tilde{\lambda}_{1, \varepsilon}(v, D_v)] f, f)_{L^2} \right|,$$

where

$$\tilde{\lambda}_{1, \varepsilon}(v, D_v) = [\tau + v \cdot \xi, \lambda_{1, \varepsilon}(v, D_v)].$$

we claim

$$\varepsilon^{-1}a^{-1}(v)[\tau + v \cdot \xi, \lambda_{1, \varepsilon}(v, D_v)] \in \text{op}_b \left( S \langle \langle \eta \rangle^{4\varepsilon}, |dv|^2 + \langle \eta \rangle^{-2} |d\eta|^2 \rangle \right).$$

Indeed, the symbol of the above operator is

$$-\frac{1}{2i\pi}e^{-\varepsilon a^{-1}(v)\xi} \cdot \partial_\phi \lambda_{1, \varepsilon}(v, \eta),$$

which belongs to $S \langle \langle \eta \rangle^{4\varepsilon}, |dv|^2 + \langle \eta \rangle^{-2} |d\eta|^2 \rangle$ uniformly with respect to $\xi$ and $\varepsilon$, since by (3.18)

$$\partial_\phi \lambda_{1, \varepsilon} \in S \langle \langle \eta \rangle^{2\varepsilon-1}, |dv|^2 + \langle \eta \rangle^{-2} |d\eta|^2 \rangle,$$

and by (2.12)

$$\left| \partial^\alpha \left( e^{-\varepsilon a^{-1}(v)} \langle \xi \rangle \right) \right| \lesssim e^{-\varepsilon a^{-1}(v)} \langle \xi \rangle \lesssim \langle \eta \rangle^{2\varepsilon+1}$$

on the support of $\lambda_{1, \varepsilon}$. Thus writing

$$a^{-1}(v)[\tau + v \cdot \xi, \lambda_{1, \varepsilon}(v, D_v)] = \varepsilon \langle D_v \rangle^{2\varepsilon} \langle D_v \rangle^{-2\varepsilon} e^{-\varepsilon a^{-1}(v)[\tau + v \cdot \xi, \lambda_{1, \varepsilon}(v, D_v)]} \langle D_v \rangle^{-2\varepsilon} \langle D_v \rangle^{2\varepsilon},$$

we have, by the $L^2$ continuity theorem in the class $S \{1, |dv|^2 + \langle \eta \rangle^{-2} |d\eta|^2 \}$,

$$\left| (a(v)a^{-1}(v)[\tau + v \cdot \xi, \lambda_{1, \varepsilon}(v, D_v)] a(v)f, f)_{L^2} \right| \lesssim \varepsilon \left( \langle D_v \rangle^{2\varepsilon} a(v) f \right)_{L^2}^2.$$
This gives the desired upper bound for the first term in (3.21). As for the second term, we may write, by (2.19) and (3.22),

$$\left[ a, a^{-1}(v) [\tau + v \cdot \xi, \lambda_{1,\varepsilon}(v, D_v)] \right] = \varepsilon \left[ a, a^{-1}(v) [\tau + v \cdot \xi, \lambda_{1,\varepsilon}(v, D_v)] \right]$$

$$= \varepsilon R(v, D_v) + \varepsilon \tau(v, D_v)$$

with

$$R(v, \eta) \in S \left( (v)^{\tau+\gamma} (\eta)^{\delta}, |dv|^2 + |d\eta|^2 \right), \quad r(v, \eta) \in S \left( (v)^{\tau+\gamma} (\eta)^{\delta}, |dv|^2 + |d\eta|^2 \right).$$

This implies

$$\left| (a(v) \left[ a, a^{-1}(v) \right] f, f \right|_{L^2} \right| \lesssim \varepsilon \left| (a(v) R(v, D_v) f, f \right|_{L^2} \right| + \varepsilon \left| (a(v) r(v, D_v) f, f \right|_{L^2} \right|$$

$$\lesssim \varepsilon \left| (D^2_v a(v) f \right|_{L^2} + (D^2_v)^{-\delta} R(v, D_v) f \right|_{L^2} \right| + \varepsilon \left| (D^2_v a(v) f \right|_{L^2} \right| (D^2_v)^{-\delta} r(v, D_v) f \right|_{L^2} \right|$$

$$\lesssim \varepsilon \left| (D^2_v a(v) f \right|_{L^2}^2 + C_r \left( \| \overline{P} f \|_{L^2}^2 + \| f \|_{L^2}^2 \right),$$

where the last inequality holds because of (3.1). We have proven (3.20).

**Step 3:** Let \( \lambda_{2,\varepsilon}(v, \eta) \) be given in (3.17). In this step we show

$$\left| (a(v) \left[ \tau + v \cdot \xi, \lambda_{2,\varepsilon}(v, D_v) \right] f, f \right|_{L^2} \right| \lesssim \varepsilon \left| (\tau + v \cdot \xi) f \right|_{L^2}^2 + C_r \left( \| \overline{P} f \|_{L^2}^2 + \| f \|_{L^2}^2 \right). \quad (3.23)$$

Using the notation

$$\tilde{\lambda}_{2,\varepsilon}(v, D_v) = a \overline{\tau} \lambda_{2,\varepsilon}(v, D_v),$$

and writing

$$a(v) \left[ \tau + v \cdot \xi, \lambda_{2,\varepsilon}(v, D_v) \right] = a \overline{\tau} \left[ \tau + v \cdot \xi, a \overline{\tau} \lambda_{2,\varepsilon}(v, D_v) \right]$$

$$= (\tau + v \cdot \xi) a \overline{\tau} \tilde{\lambda}_{2,\varepsilon}(v, D_v) - a \overline{\tau} \tilde{\lambda}_{2,\varepsilon}(v, D_v) (\tau + v \cdot \xi),$$

we have, with \( \tilde{\lambda}_{2,\varepsilon}^*(v, D_v) \) denoting the adjoint of \( \tilde{\lambda}_{2,\varepsilon}(v, D_v) \),

$$\left| (a(v) \left[ \tau + v \cdot \xi, \lambda_{2,\varepsilon}(v, D_v) \right] f, f \right|_{L^2} \right| \lesssim \varepsilon \left| (\tau + v \cdot \xi) f \right|_{L^2}^2 + \| \tau + v \cdot \xi \|_{L^2} \left( \left| a \overline{\tau} \tilde{\lambda}_{2,\varepsilon}(v, D_v) f \right|_{L^2} \right)+ \| \tilde{\lambda}_{2,\varepsilon}^*(v, D_v) a \overline{\tau} \tilde{\lambda}_{2,\varepsilon} \overline{\tau} \tau f \right|_{L^2}.$$
where the last inequality holds because by (2.20) we have
\[
\left\| \epsilon \frac{2}{\tau + v \cdot \xi} \langle \xi \rangle^{-\frac{3}{2}} \tilde{\lambda}_{2,e}(v, D_v, a^{\frac{2}{\tau + v \cdot \xi}} (v)) \right\|_{L^2}^2 \\
\lesssim \left\| a^{\frac{2}{\tau + v \cdot \xi}} (v) \right\|_{L^2}^2 + \left\| \epsilon \frac{2}{\tau + v \cdot \xi} \langle \xi \rangle^{-\frac{3}{2}} \right\|_{L^2}^2.
\]
Moreover, we have
\[
\epsilon \frac{2}{\tau + v \cdot \xi} \langle \xi \rangle^{-\frac{3}{2}} \tilde{\lambda}_{2,e}(v, D_v, a^{\frac{2}{\tau + v \cdot \xi}} (v)) \in S \left( 1, |d v|^2 + |\eta|^{-2} |d \eta|^2 \right),
\]
which gives
\[
\left\| \tilde{\lambda}_{2,e}(v, D_v, a^{\frac{2}{\tau + v \cdot \xi}} (v)) \right\|_{L^2}^2 = \epsilon^{-\frac{2}{\tau + v \cdot \xi}} \left\| \epsilon \frac{2}{\tau + v \cdot \xi} \langle \xi \rangle^{-\frac{3}{2}} \tilde{\lambda}_{2,e}(v, D_v, a^{\frac{2}{\tau + v \cdot \xi}} (v)) \right\|_{L^2}^2 \\
\lesssim C \epsilon \left\| a^{\frac{2}{\tau + v \cdot \xi}} (v) \right\|_{L^2}^2.
\]
Combining these inequalities, we obtain the desired estimate (3.24), since by (1.2) and (3.3), one has
\[
\left\| (v) a^{\frac{2}{\tau + v \cdot \xi}} (v) \right\|_{L^2}^2 \lesssim \left\| a^{\frac{2}{\tau + v \cdot \xi}} (v) \right\|_{L^2}^2 \lesssim \| \tilde{\Phi} f \|_{L^2}^2 + \| f \|_{L^2}^2.
\]

**Step 4:** Now we are ready to treat the second term on the left-hand side of (3.15), and prove
\[
\left\| a(v) (D_v)^{2s} f \right\|_{L^2}^2 = \| \tilde{\Phi} f \|_{L^2}^2 + \| f \|_{L^2}^2.
\]
In fact the relation
\[
\text{Re} \left( \tilde{\Phi} f, a(v) (D_v)^{2s} f \right)_{L^2} = \text{Re} \left( i (\tau + v \cdot \xi) f, a(v) (D_v)^{2s} f \right)_{L^2} \\
+ \text{Re} \left( a(D_v)^{2s} f, a(v) (D_v)^{2s} f \right)_{L^2} + \text{Re} \left( b(v) f, a(v) (D_v)^{2s} f \right)_{L^2}
\]
implies
\[
\left\| a(v) (D_v)^{2s} f \right\|_{L^2}^2 \lesssim \| \tilde{\Phi} f \|_{L^2}^2 + \| f \|_{L^2}^2 - \text{Re} \left( i (\tau + v \cdot \xi) f, a(v) (D_v)^{2s} f \right)_{L^2},
\]
since
\[
\| b(v) f \|_{L^2}^2 \lesssim \| \tilde{\Phi} f \|_{L^2}^2 + \| f \|_{L^2}^2
\]
due to (1.2) and (3.1). Moreover we have
\[
- \text{Re} \left( i (\tau + v \cdot \xi) f, a(v) (D_v)^{2s} f \right)_{L^2} \\
= \frac{i}{2} \left\| (a(v) \left[ \tau + v \cdot \xi, (D_v)^{2s} \right] f, f \right\|_{L^2} + \frac{i}{2} \left\| (\tau + v \cdot \xi) [a(v), (D_v)^{2s}] f, f \right\|_{L^2} \\
\lesssim \left\| (a(v) \left[ \tau + v \cdot \xi, (D_v)^{2s} \right] f, f \right\|_{L^2} + \epsilon \left\| (\tau + v \cdot \xi) f \right\|_{L^2}^2 + C \epsilon \left\| [a(v), (D_v)^{2s}] f \right\|_{L^2}^2.
\]
On the other hand, note that
\[
(D_v)^{2s} = \lambda_{1,e}(v, D_v) + \lambda_{2,e}(v, D_v),
\]
with \( \lambda_{1,e}, \lambda_{2,e} \) defined in (3.16) and (3.17). Then it follows from (3.20) and (2.33) that
\[
\left\| a(v) \left[ \tau + v \cdot \xi, (D_v)^{2s} \right] f, f \right\|_{L^2} \\
\lesssim \left\| a(v) \left[ \tau + v \cdot \xi, \lambda_{1,e}(v, D_v) \right] f, f \right\|_{L^2} + \left\| a(v) \left[ \tau + v \cdot \xi, \lambda_{2,e}(v, D_v) \right] f, f \right\|_{L^2} \\
\lesssim \epsilon \left\| (D_v)^{2s} a(v) f \right\|_{L^2}^2 + \epsilon \left\| (\tau + v \cdot \xi) f \right\|_{L^2}^2 + C \epsilon \left( \| \tilde{\Phi} f \|_{L^2}^2 + \| f \|_{L^2}^2 \right).
\]
Combining these inequalities, we conclude
\[
\|a(v) \langle D_v \rangle^{2s} f \|_{L^2}^2 \\
\lesssim \varepsilon \|\langle D_v \rangle^{2s} a(v) f \|_{L^2}^2 + \varepsilon \|\tau + v \cdot \xi \|_{L^2} + C_\varepsilon \left( \|\tilde{\mathcal{P}} f \|_{L^2}^2 + \|f\|_{L^2}^2 + \|\langle a(v) , \langle D_v \rangle^{2s} \rangle f \|_{L^2}^2 \right)
\]
\[
\lesssim \varepsilon \|a(v) \langle D_v \rangle^{2s} f \|_{L^2}^2 + C_\varepsilon \left( \|\tilde{\mathcal{P}} f \|_{L^2}^2 + \|f\|_{L^2}^2 + \|\langle a(v) , \langle D_v \rangle^{2s} \rangle f \|_{L^2}^2 \right),
\]
where the last inequality holds because
\[
\|\tau + v \cdot \xi \|_{L^2} \lesssim \|\tilde{\mathcal{P}} f \|_{L^2} + \|b(v) f \|_{L^2} + \|a \langle D_v \rangle^{2s} f \|_{L^2},
\]
which gives the desired estimate (3.25), since by (2.11) and (3.1), one has
\[
\|a(v) \langle D_v \rangle^{2s} f \|_{L^2}^2 \lesssim \|\tilde{\mathcal{P}} f \|_{L^2}^2 + \|f\|_{L^2}^2.
\]

**Step 5:** Now it remains to treat the first term on the left-hand side of (3.15). By computation, we have
\[
\langle v \rangle^{-\frac{2s}{2s-n}} a \frac{1}{\tau + v \cdot \xi} \langle \tau + v \cdot \xi \rangle^{\frac{2s}{2s-n}} \lesssim \langle v \rangle^{-\frac{2s}{2s-n}} a \frac{1}{\tau + v \cdot \xi} \langle \tau + v \cdot \xi \rangle^{\frac{2s}{2s-n}} + \langle v \rangle^{-\frac{2s}{2s-n}} a \frac{1}{\tau + v \cdot \xi} \langle v \cdot \xi \rangle^{\frac{2s}{2s-n}}
\]
\[
\lesssim \langle v \rangle^{-\frac{2s}{2s-n}} a \frac{1}{\tau + v \cdot \xi} \langle \tau + v \cdot \xi \rangle^{\frac{2s}{2s-n}} + \langle v \rangle^{-\frac{2s}{2s-n}} a \frac{1}{\tau + v \cdot \xi} \langle v \cdot \xi \rangle^{\frac{2s}{2s-n}}
\]
\[
\lesssim \langle v \rangle^{-2s} a(v) + (|\tau + v \cdot \xi| + 1) a \frac{1}{\tau + v \cdot \xi} \langle v \cdot \xi \rangle^{\frac{2s}{2s-n}},
\]
where the last inequality follows from Young’s inequality:
\[
\langle v \rangle^{-\frac{2s}{2s-n}} a \frac{1}{\tau + v \cdot \xi} \langle \tau + v \cdot \xi \rangle^{\frac{2s}{2s-n}} \leq \left( \langle v \rangle^{-\frac{2s}{2s-n}} a \frac{1}{\tau + v \cdot \xi} \langle \tau + v \cdot \xi \rangle^{\frac{2s}{2s-n}} \right)^{1+2s} + \frac{2s}{1+2s} \left( (\tau + v \cdot \xi)^{\frac{2s}{2s-n}} \right)^{(1+2s)/(2s)}.
\]
As a result we have, using the relation (3.26),
\[
\|\langle v \rangle^{-\frac{2s}{2s-n}} a \frac{1}{\tau + v \cdot \xi} \langle \tau + v \cdot \xi \rangle^{\frac{2s}{2s-n}} f \|_{L^2} \lesssim \|\tau + v \cdot \xi \|_{L^2} + \|\langle v \rangle^{-2s} a(v) f \|_{L^2} + \|f\|_{L^2} + \|a \frac{1}{\tau + v \cdot \xi} \langle v \cdot \xi \rangle^{\frac{2s}{2s-n}} f \|_{L^2}
\]
\[
\lesssim \|\tilde{\mathcal{P}} f \|_{L^2} + \|a(v) \langle D_v \rangle^{2s} f \|_{L^2} + \|f\|_{L^2} + \|a \frac{1}{\tau + v \cdot \xi} \langle v \cdot \xi \rangle^{\frac{2s}{2s-n}} f \|_{L^2}
\]
\[
\lesssim \|\tilde{\mathcal{P}} f \|_{L^2} + \|f\|_{L^2},
\]
where the last inequality follows from (3.1), (3.3) and (3.25). The proof of Proposition 3.5 is completed. \(\Box\)

**A Appendix**

For the sake of completeness, we collect here some results concerned with the composition formulas and the estimates on commutators, and we refer to Theorem 2.3.19, Theorem 1.1.20, Lemma 4.1.5, Theorem 2.3.8 of [27].

**Theorem A.1** Let \( j = 1, 2 \) and \( p_j \in S \left( m_j, \langle v \rangle^{-2} |dv|^2 + \langle \eta \rangle^{-2} |d\eta|^2 \right) \) with \( m_1 = \langle v \rangle^x \) and \( m_2 = \langle \eta \rangle^y \). Then
\[
p_1 (v, D_v) p_2 (v, D_v) = (p_1 \circ p_2) (v, D_v),
\]
where
\[ p_1 \ast p_2 = p_1 p_2 + r \]
with \( r \in S ( (v)^{\delta-1} \langle \eta \rangle^{\delta-1}, (v)^{-2} |dv|^2 + (\eta)^{-2} |d\eta|^2 ) \). In particular the symbol of the commutator
\[ [p_1 (v, D_v), p_2 (v, D_v)] \]
belongs to \( S ( (v)^{\delta-1} \langle \eta \rangle^{\delta-1}, (v)^{-2} |dv|^2 + (\eta)^{-2} |d\eta|^2 ) \).

**Theorem A.2** Let \( m_j \) be real numbers and let \( p_j \in S ( (\eta)^{m_j}, |dv|^2 + (\eta)^{-2} |d\eta|^2 ) \). Then
\[ p_1 (v, D_v) p_2 (v, D_v) = (p_1 \ast p_2) (v, D_v), \]
where
\[ p_1 \ast p_2 = p_1 p_2 + (D_\eta p_1) \partial_x p_2 + r \]
with
\[ r (v, \eta) = \int_0^1 (1 - \theta)^2 e^{2i\pi \theta D_x} (2i\pi D_x \cdot D_\xi)^2 (p_1 (v, \xi) p_2 (\xi, \eta)) d\theta \]
\[ \bigg|_{\xi = v, \xi = \eta}. \]

**Proposition A.3** Let \( m \) be real numbers and let \( p \in S ( (\eta)^{m}, |dv|^2 + (\eta)^{-2} |d\eta|^2 ) \). Then
\[ J^0 p (v, \eta) \]
\[ \text{def} = (e^{2i\pi \partial_x} D_\xi) p (v, \eta) \]
belongs to \( S ( (\eta)^{m}, |dv|^2 + (\eta)^{-2} |d\eta|^2 ) \).

**Theorem A.4** Let \( t_1, t_2 \geq 0 \) be given and let \( p_j \in S (m_j, g_j) \) with \( g_1 = (v)^{-t_1} |dv|^2 + (\eta)^{-t_2} |d\eta|^2 \), \( g_2 = |dv|^2 + (\eta)^{-t_2} |d\eta|^2 \) and \( m_1, m_2 \) two admissible weights. Define \( \Lambda_{12} \) by
\[ \Lambda_{12} (X) = \inf_T \left( g_1^\alpha (T) / g_2^\alpha (T) \right)^{1/2}. \]
Then
\[ p_1^w \ast p_2^w = (p_1 \ast p_2)^{w'}. \]
where
\[ p_1 \ast p_2 = p_1 p_2 + r \]
with \( r \in S (m_1 m_2 \Lambda_{12}^{-1}, 1/2 (g_1 + g_2)) \). In particular the symbol of the commutator
\[ [p_1^w, p_2^w] \]
belongs to \( S (m_1 m_2 \Lambda_{12}^{-1}, 1/2 (g_1 + g_2)) \).

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