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L^2-regularity of kinetic equations with external potential

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Abstract

In this paper, we study a class of kinetic equations with a potential term. Making use of the multiplier method, we obtain hypoelliptic regularity of the solution u on space variable x and the potential $V(x)$, under some assumptions on the velocity variable y and the potential $V(x)$.

Keywords: Kinetic equation; Potential term; Hypoelliptic regularity

1. Introduction

This paper is motivated by the study of the kinetic equations. We are interested in the following equation

$$\partial_t u + y \cdot \nabla_x u - \nabla_x V(x) \cdot \nabla_y u = f(t, x, y), \tag{1.1}$$

where $t \in \mathbb{R}$ is the time variable, $x \in \mathbb{R}^n$ is the space variable, $y \in \mathbb{R}^n$ denotes the velocity variable and $V(x)$ is a potential defined in the whole space $\mathbb{R}^n_x$. Averaging lemmas arise in the study of regularity of solutions to the above transport equations. The present paper shows how techniques from microlocal analysis can be used to prove the regularity properties and to establish their sharpness. This type of questions concerns Fokker–Planck equations, Landau equations,

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Boltzmann equations, and has been studied by Desvillettes–Villani, Helffer–Nier and other authors (see [1–7]). Without potential, Bouchut [1] got hypoelliptic regularity by three types of proofs. One uses Fourier transform, another one depends on Hörmander's commutators, and the last one applies a characteristics commutator. However, to get the results, one needs some assumption and the authors conjecture that the assumption is not necessary in the general case. We want to remove this extra condition. As far as the potential term is concerned, some conditions on $V(x)$ had been considered in [3,6,7,10]. For the model (1.1), the methods in [1] are not suitable for use. Fortunately, inspired by the previous works of Hérau–Li in [7], we obtain the hypoelliptic estimates for the solution of the equation (1.1) which can be seen as an improvement of Bouchut's work in [1].

For convenience, we write $D_t = -i\partial_t$, $D_x = -i\nabla_x$, $D_y = -i\nabla_y$, $\langle y\rangle = (1+|y|^2)^{1/2}$. Then our main results are as follows.

First, we give some results on equations without potential term. Then we can get a direct compare with the result of [1]. And we can see the advantage of the multiplier method.

Theorem 1.1. *Suppose $f(t,x,y) \in L^2(\mathbb{R}^{2n+1}_{t,x,y})$ and $u(t,x,y)$ is a solution of the following equation,*

$$\partial_t u + y\cdot\nabla_x u = f(t,x,y). \tag{1.2}$$

If $\langle D_y\rangle^p u$, $\langle D_y\rangle^q f \in L^2(\mathbb{R}^{2n+1}_{t,x,y})$ with $p,q\in\mathbb{R}^+$ satisfying $q\neq 1+p$. Then $\langle D_x\rangle^{p/(1+p-q)} u \in L^2(\mathbb{R}^{2n+1}_{t,x,y})$ and

$$\|\langle D_x\rangle^{\frac{p}{1+p-q}} u\|_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} \lesssim \|\langle D_y\rangle^p u\|_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} + \|\langle D_y\rangle^q f\|_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}. \tag{1.3}$$

Remark 1.1. This improves the regularity index in [1].

Remark 1.2. In our results, if we take $p = q+1$ and $p\to+\infty$, then $p/(p+1-q)\to+\infty$. This means that for all $\gamma\in\mathbb{R}^+$, there exist p,q such that if $\langle D_y\rangle^p u$, $\langle D_y\rangle^q f\in L^2(\mathbb{R}^{2n+1}_{t,x,y})$, we can have $\langle D_x\rangle^\gamma u\in L^2(\mathbb{R}^{2n+1}_{t,x,y})$. However, in [1], since the condition $0\le q\le 1$, one can only obtain $\langle D_x\rangle u\in L^2(\mathbb{R}^{2n+1}_{t,x,y})$ even with $\langle D_y\rangle^{\max\{p,q\}}u$, $\langle D_y\rangle^q f\in L^2(\mathbb{R}^{2n+1}_{t,x,y})$ for any p,q. And one can not have $\langle D_x\rangle^\gamma u\in L^2(\mathbb{R}^{2n+1}_{t,x,y})$ for any $\gamma>1$.

In fact, the multiplier method can also be suitable for the kinetic equations (1.1).

Theorem 1.2. *Let $V(x)\in C^2(\mathbb{R}^n,\mathbb{R})$ be a real-valued function satisfying that*

$$\forall|\beta|=2,\ \exists C_\beta>0\ s.t.\ \forall x\in\mathbb{R}^n,\ |\partial_x^\beta V(x)|\le C_\beta\langle\nabla_x V(x)\rangle^\gamma. \tag{1.4}$$

Suppose $f(t,x,y)\in L^2(\mathbb{R}^{2n+1}_{t,x,y})$ and $u(t,x,y)$ is a solution of the equation (1.1). If $\langle y\rangle^r u$, $\langle y\rangle^s f$, $\langle D_y\rangle^p u\in L^2(\mathbb{R}^{2n+1}_{t,x,y})$ with $r,s,p\in\mathbb{R}^+$, $\gamma\in\mathbb{R}$ satisfying

$$\min\{r,p\}\ge s+1,\quad 0\le\gamma\le\frac{2r}{1+r-s}\min\{1-\frac{s+1}{r},1-\frac{s+1}{p}\}, \tag{1.5}$$

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then $\langle \nabla_x V(x)\rangle^{r/(1+r-s)} u \in L^2(\mathbb{R}^{2n+1}_{t,x,y})$ *and*

$$\|\langle \nabla_x V(x)\rangle^{\frac{r}{1+r-s}} u\|_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} \lesssim \|\langle y\rangle^r u\|_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} + \|\langle y\rangle^s f\|_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} + \|\langle D_y\rangle^p u\|_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}. \tag{1.6}$$

Theorem 1.3. *Suppose the conditions in Theorem 1.2. Furthermore, if* $\langle D_y\rangle^q f \in L^2(\mathbb{R}^{2n+1}_{t,x,y})$ *and*

$$\min\{r, p\} \geq q+1, \quad 0 \leq \gamma \leq \frac{2r}{1+r-s}\min\{1-\frac{q+1}{r}, 1-\frac{q+1}{p}\}, \tag{1.7}$$

then we have:

(i) $\langle D_x\rangle^{p/(1+p-q)} u \in L^2(\mathbb{R}^{2n+1}_{t,x,y})$ *and*

$$\|\langle D_x\rangle^{\frac{p}{1+p-q}} u\|_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} \lesssim \|\langle D_y\rangle^p u\|_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} + \|\langle D_y\rangle^q f\|_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} + \|\langle y\rangle^r u\|_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} + \|\langle y\rangle^s f\|_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}. \tag{1.8}$$

(ii) *If* $0 < q < 1$, $r \leq \frac{p}{q}(s-1)$, *then* $\langle y\rangle^{-p/(1+p-q)}|D_t|^{p/(1+p-q)} u \in L^2(\mathbb{R}^{2n+1}_{t,x,y})$ *and*

$$\|\langle y\rangle^{-\frac{p}{1+p-q}}|D_t|^{\frac{p}{1+p-q}} u\|_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} \lesssim \|\langle D_y\rangle^p u\|_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} + \|\langle D_y\rangle^q f\|_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} + \|\langle y\rangle^r u\|_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} + \|\langle y\rangle^s f\|_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}. \tag{1.9}$$

The plan of this paper is as follows. In the second section, we use the multiplier method to prove Theorem 1.1. Then we consider the equations with potential term in Section 3 and provide the proof of Theorem 1.2 and Theorem 1.3. At last, we give some facts on symbolic calculus, especially on Wick and Weyl quantization in Appendix.

2. Proof of Theorem 1.1

Given a symbol p, we use p^{Wick} and p^w to denote the Wick and Weyl quantization of p in the variables (y, η). Consider x, ξ, t, τ as parameters. Write $P = i(D_t + y \cdot D_x)$. Throughout this paper, we say $A \lesssim B$ if there exists a positive constant C such that $A \leq CB$, and similarly for $A \gtrsim B$. While the notation $A \approx B$ means both $A \lesssim B$ and $A \gtrsim B$ hold.

The proof of Theorem 1.1. For $q > 1+p$, we have $\frac{p}{1+p-q} < 0$. Then (1.3) is obvious.
For $q < 1+p$, we define the symbols Q and g_1 by

$$Q(y,\eta) = D_t + y\cdot\xi, \quad g_1(y,\eta) = \frac{\xi\cdot\eta}{\langle\xi\rangle^{2(1-q)/(1+p-q)}}\chi(\frac{\langle\eta\rangle^{p+1-q}}{\langle\xi\rangle}), \tag{2.1}$$

where $\chi \in C_0^\infty(\mathbb{R}; [0,1])$ such that $\chi = 1$ in $[-1,1]$ and supp $\chi \subset [-2,2]$.

From (1.2) and (2.1), we see

$$Re(iQ\hat{u}, g_1^{Wick}\hat{u})_{L^2(\mathbb{R}^{n+1}_{t,y})} = Re(\hat{f}, g_1^{Wick}\hat{u})_{L^2(\mathbb{R}^{n+1}_{t,y})}, \tag{2.2}$$

where $\hat{f}$ is the Fourier transform of f in x variable. Obverse that the symbol Q is a first order polynomial in y, η and this implies that $Q^{Wick} = Q^w = Q$ (this will be proved in Appendix). Then

$$Re(iQ\hat{u}, g_1^{Wick}\hat{u})_{L^2(\mathbb{R}^{n+1}_{t,y})} = \frac{1}{4\pi}(\{g_1, D_t + y\cdot\xi\}^{Wick}\hat{u}, \hat{u})_{L^2(\mathbb{R}^{n+1}_{t,y})}. \tag{2.3}$$

Here $\{a_1, a_2\}$ denotes the Poisson bracket of the symbols a_1 and a_2 which is defined by

$$\{a_1, a_2\} = \sum_{j=1}^{n} \frac{\partial a_1}{\partial \eta_j}\frac{\partial a_2}{\partial y_j} - \frac{\partial a_1}{\partial y_j}\frac{\partial a_2}{\partial \eta_j}.$$

Therefore,

$$\begin{aligned}
\{g_1, D_t + y\cdot\xi\} &= \frac{|\xi|^2}{\langle\xi\rangle^{2(1-q)/(1+p-q)}}\chi + \frac{\xi\cdot\eta}{\langle\xi\rangle^{2(1-q)/(1+p-q)}}\chi'\frac{(p+1-q)\langle\eta\rangle^{p-q-1}\eta}{\langle\xi\rangle}\cdot\xi \\
&= \frac{|\xi|^2+1-1}{\langle\xi\rangle^{2(1-q)/(1+p-q)}}\chi + \frac{(p+1-q)|\xi\cdot\eta|^2}{\langle\xi\rangle^{1+2(1-q)/(1+p-q)}}\langle\eta\rangle^{p-q-1}\chi' \\
&\ge \langle\xi\rangle^{2p/(1+p-q)} - \langle\xi\rangle^{2p/(1+p-q)}(1-\chi) - \langle\xi\rangle^{-2(1-q)/(1+p-q)} \\
&\ge \langle\xi\rangle^{2p/(1+p-q)} - \langle\eta\rangle^{2p} - \langle\xi\rangle^{-2(1-q)/(1+p-q)}.
\end{aligned} \tag{2.4}$$

Combining (2.3), (2.4) and using the positivity of the Wick quantization, we have the following estimate,

$$((\langle\xi\rangle^{2p/(1+p-q)})^{Wick}\hat{u}, \hat{u})_{L^2(\mathbb{R}^{n+1}_{t,y})} \le I_1 + I_2 + I_3, \tag{2.5}$$

where

$$\begin{cases}
I_1 = ((\langle\xi\rangle^{-2(1-q)/(1+p-q)})^{Wick}\hat{u}, \hat{u})_{L^2(\mathbb{R}^{n+1}_{t,y})}, \\
I_2 = ((\langle\eta\rangle^{2p})^{Wick}\hat{u}, \hat{u})_{L^2(\mathbb{R}^{n+1}_{t,y})}, \\
I_3 = |(\hat{f}, g_1^{Wick}\hat{u})_{L^2(\mathbb{R}^{n+1}_{t,y})}|.
\end{cases} \tag{2.6}$$

Note that the Wick quantization is about the variables (y, η), then (2.5) implies

$$(\langle\xi\rangle^{2p/(1+p-q)}\hat{u}, \hat{u})_{L^2(\mathbb{R}^{n+1}_{t,y})} \le I_1 + I_2 + I_3. \tag{2.7}$$

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First, we calculate I_1. Since $q < 1 + p$, then for all $\varepsilon > 0$ we have

$$\begin{aligned} I_1 &= \|\langle\xi\rangle^{-(1-q)/(1+p-q)}\hat{u}\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})} \\ &\le \varepsilon(\langle\xi\rangle^{2p/(1+p-q)}\hat{u},\hat{u})_{L^2(\mathbb{R}^{n+1}_{t,y})} + C_\varepsilon\|\hat{u}\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})}. \end{aligned} \tag{2.8}$$

Next, we estimate I_2. Let $A = \langle D_y\rangle^p$, then $A = A^*$, $A^{-1} = \langle D_y\rangle^{-p}$ and we have

$$\begin{aligned} I_2 &= (AA^{-1}(\langle\eta\rangle^{2p})^{Wick}A^{-1}A\hat{u},\hat{u})_{L^2(\mathbb{R}^{n+1}_{t,y})} \\ &= (A^{-1}(\langle\eta\rangle^{2p})^{Wick}A^{-1}A\hat{u},A\hat{u})_{L^2(\mathbb{R}^{n+1}_{t,y})} \\ &\lesssim \|A\hat{u}\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})} = \|\langle D_y\rangle^p\hat{u}\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})}, \end{aligned} \tag{2.9}$$

where we use the fact that the symbol of $A^{-1}(\langle\eta\rangle^{2p})^{Wick}A^{-1}$ belongs to $S(1, |dy|^2 + |d\eta|^2)$.

Finally, we evaluate I_3. For all $\varepsilon > 0$,

$$\begin{aligned} I_3 &= |(\langle D_y\rangle^q\hat{f},\langle D_y\rangle^{-q}g_1^{Wick}\hat{u})_{L^2(\mathbb{R}^{n+1}_{t,y})}| \le C_\varepsilon\|\langle D_y\rangle^q\hat{f}\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})} + \varepsilon\|\langle D_y\rangle^{-q}g_1^{Wick}\hat{u}\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})} \\ &\le C_\varepsilon\|\langle D_y\rangle^q\hat{f}\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})} \\ &\quad + \varepsilon(\langle\xi\rangle^{2p/(1+p-q)}\hat{u}, \Big(\frac{\langle\eta\rangle^{2-2q}}{\langle\xi\rangle^{2(1-q)/(1+p-q)}}\chi^2(\frac{\langle\eta\rangle^{p+1-q}}{\langle\xi\rangle})\Big)^{Wick}\hat{u})_{L^2(\mathbb{R}^{n+1}_{t,y})} \\ &\le C_\varepsilon\|\langle D_y\rangle^q\hat{f}\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})} + \varepsilon(\langle\xi\rangle^{2p/(1+p-q)}\hat{u},\hat{u})_{L^2(\mathbb{R}^{n+1}_{t,y})}. \end{aligned} \tag{2.10}$$

Combining (2.7), (2.8), (2.9) and (2.10), we can see

$$\langle\xi\rangle^{2p/(1+p-q)}\|\hat{u}\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})} \lesssim \|\langle D_y\rangle^p\hat{u}\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})} + \|\langle D_y\rangle^q\hat{f}\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})}. \tag{2.11}$$

Integrating both sides of (2.11) with respect to ξ, we gain

$$\|\langle\xi\rangle^{p/(1+p-q)}\hat{u}\|^2_{L^2(\mathbb{R}^{2n+1}_{t,\xi,y})} \lesssim \|\langle D_y\rangle^p\hat{u}\|^2_{L^2(\mathbb{R}^{2n+1}_{t,\xi,y})} + \|\langle D_y\rangle^q\hat{f}\|^2_{L^2(\mathbb{R}^{2n+1}_{t,\xi,y})}. \tag{2.12}$$

By Plancherel's formula, we have

$$\|\langle D_x\rangle^{p/(1+p-q)}u\|_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} \lesssim \|\langle D_y\rangle^p u\|_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} + \|\langle D_y\rangle^q f\|_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}. \quad \square \tag{2.13}$$

3. Hypoelliptic regularity with external potential

In this section, we consider the kinetic equations with external potential, and prove the hypoelliptic regularity of (1.1). First, we consider the estimate on potential term $V(x)$.

Proposition 3.1. *For fixed $r, s \in \mathbb{R}^+$, assume that $u, g \in L^2(\mathbb{R}^{2n+1})$ satisfy the equation*

$$\partial_t u + i(y \cdot \xi - \nabla_x V(x) \cdot D_y)u = g. \tag{3.1}$$

If $\langle y\rangle^r u \in L^2(\mathbb{R}^{2n+1})$, $\langle y\rangle^s g \in L^2(\mathbb{R}^{2n+1})$, $s \neq 1+r$, then

$$\langle \nabla_x V(x)\rangle^{\frac{2r}{1+r-s}} \|u\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})} \lesssim \|\langle y\rangle^r u\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})} + \|\langle y\rangle^s g\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})}. \tag{3.2}$$

Proof. Since for $s > 1+r$ the result is obvious, then we only need to consider the case of $s < 1+r$. Suppose Q_X and g_2 are the symbols defined by

$$Q_X(y,\eta) = D_t + y\cdot\xi - \nabla_x V(x)\cdot\eta, \quad g_2(y,\eta) = \frac{\nabla_x V(x)\cdot y}{\langle \nabla_x V(x)\rangle^{\frac{2-2s}{1+r-s}}}\chi\Big(\frac{\langle y\rangle^{1+r-s}}{\langle \nabla_x V(x)\rangle}\Big), \tag{3.3}$$

where $\chi \in C_0^\infty(\mathbb{R}; [0,1])$ such that $\chi = 1$ in $[-1,1]$ and $\operatorname{supp}\chi \subset [-2,2]$. Then $Q_X^{Wick} = Q_X^w = D_t + y\cdot\xi - \nabla_x V(x)\cdot D_y$.

From (3.1) and (3.3), we know

$$\begin{aligned} Re(g, g_2^{Wick} u)_{L^2(\mathbb{R}^{n+1}_{t,y})} &= Re(iQ_X^{Wick} u, g_2^{Wick} u)_{L^2(\mathbb{R}^{n+1}_{t,y})} \\ &= \frac{1}{4\pi}(\{g_2, D_t + y\cdot\xi - \nabla_x V(x)\cdot\eta\}^{Wick} u, u)_{L^2(\mathbb{R}^{n+1}_{t,y})}. \end{aligned}$$

Similar to (2.4), we can also gain

$$\begin{aligned} &\{g_2, D_t + y\cdot\xi - \nabla_x V(x)\cdot\eta\} \\ &= \frac{|\nabla_x V(x)|^2}{\langle \nabla_x V(x)\rangle^{\frac{2-2s}{1+r-s}}}\chi + \frac{\nabla_x V(x)\cdot y}{\langle \nabla_x V(x)\rangle^{\frac{2-2s}{1+r-s}}}\chi' \frac{(r+1-s)\langle y\rangle^{r-s-1} y}{\langle \nabla_x V(x)\rangle}\cdot \nabla_x V(x) \\ &= \frac{|\nabla_x V(x)|^2 + 1 - 1}{\langle \nabla_x V(x)\rangle^{\frac{2-2s}{1+r-s}}}\chi + \frac{(r+1-s)|\nabla_x V(x)\cdot y|^2}{\langle \nabla_x V(x)\rangle^{\frac{3-3s+r}{1+r-s}}}\langle y\rangle^{r-s-1}\chi' \\ &\geq \langle \nabla_x V(x)\rangle^{\frac{2r}{1+r-s}} - \langle y\rangle^{2r} - \langle \nabla_x V(x)\rangle^{-\frac{2-2s}{1+r-s}}. \end{aligned}$$

Then, we have the following estimate,

$$\begin{aligned} ((\langle \nabla_x V(x)\rangle^{\frac{2r}{1+r-s}})^{Wick} u, u)_{L^2(\mathbb{R}^{n+1}_{t,y})} &\leq ((\langle y\rangle^{2r})^{Wick} u, u)_{L^2(\mathbb{R}^{n+1}_{t,y})} \\ &+ ((\langle \nabla_x V(x)\rangle^{-\frac{2-2s}{1+r-s}})^{Wick} u, u)_{L^2(\mathbb{R}^{n+1}_{t,y})} + |(g, g_2^{Wick} u)_{L^2(\mathbb{R}^{n+1}_{t,y})}|. \end{aligned} \tag{3.4}$$

Thus,

$$\begin{aligned} &(\langle \nabla_x V(x)\rangle^{\frac{2r}{1+r-s}} u, u)_{L^2(\mathbb{R}^{n+1}_{t,y})} \\ &\leq ((\langle y\rangle^{2r})^{Wick} u, u)_{L^2(\mathbb{R}^{n+1}_{t,y})} + \langle \nabla_x V(x)\rangle^{-\frac{2-2s}{1+r-s}} u, u)_{L^2(\mathbb{R}^{n+1}_{t,y})} + |(g, g_2^{Wick} u)_{L^2(\mathbb{R}^{n+1}_{t,y})}|. \end{aligned} \tag{3.5}$$

Let $B = \langle y\rangle^r$, then $B = B^*$, $B^{-1} = \langle y\rangle^{-r}$, and we have

$$\begin{aligned}((\langle y\rangle^{2r})^{Wick}u, u)_{L^2(\mathbb{R}^{n+1}_{t,y})} &= (BB^{-1}(\langle y\rangle^{2r})^{Wick}B^{-1}Bu, u)_{L^2(\mathbb{R}^{n+1}_{t,y})}\\ &= (B^{-1}(\langle y\rangle^{2r})^{Wick}B^{-1}Bu, Bu)_{L^2(\mathbb{R}^{n+1}_{t,y})}\\ &\lesssim \|Bu\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})} = \|\langle y\rangle^r u\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})}.\end{aligned} \tag{3.6}$$

For all $\varepsilon > 0$, since $s < 1 + r$, we get

$$\begin{aligned}((\langle \nabla_x V(x)\rangle^{-\frac{2-2s}{1+r-s}})^{Wick}u, u)_{L^2(\mathbb{R}^{n+1}_{t,y})} &= \|\langle \nabla_x V(x)\rangle^{-\frac{1-s}{1+r-s}}u\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})}\\ &\le \varepsilon(\langle \nabla_x V(x)\rangle^{\frac{2r}{1+r-s}}u, u)_{L^2(\mathbb{R}^{n+1}_{t,y})} + C_\varepsilon\|u\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})}.\end{aligned} \tag{3.7}$$

Next, we estimate $|(g, g_2^{Wick}u)_{L^2(\mathbb{R}^{n+1}_{t,y})}|$.

$$\begin{aligned}&|(g, g_2^{Wick}u)_{L^2(\mathbb{R}^{n+1}_{t,y})}| = |(\langle y\rangle^s g, \langle y\rangle^{-s}g_2^{Wick}u)_{L^2(\mathbb{R}^{n+1}_{t,y})}|\\ &\quad\le C_\varepsilon\|\langle y\rangle^s g\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})} + \varepsilon\|\langle y\rangle^{-s}g_2^{Wick}u\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})}\\ &\quad\le C_\varepsilon\|\langle y\rangle^s g\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})} + \varepsilon(\langle \nabla_x V(x)\rangle^{\frac{2r}{1+r-s}}u, \Big(\frac{\langle y\rangle^{2-2s}}{\langle \nabla_x V(x)\rangle^{\frac{2-2s}{1+r-s}}}\chi^2\big(\frac{\langle y\rangle^{1+r-s}}{\langle \nabla_x V(x)\rangle}\big)\Big)^{Wick}u)_{L^2(\mathbb{R}^{n+1}_{t,y})}\\ &\quad\le C_\varepsilon\|\langle y\rangle^s g\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})} + \varepsilon\|\langle \nabla_x V(x)\rangle^{\frac{r}{1+r-s}}u\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})}.\end{aligned} \tag{3.8}$$

From (3.5), (3.6), (3.7) and (3.8), we can see

$$(\langle \nabla_x V(x)\rangle^{\frac{2r}{1+r-s}}u, u)_{L^2(\mathbb{R}^{n+1}_{t,y})} \lesssim \|\langle y\rangle^r u\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})} + \|\langle y\rangle^s g\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})}. \tag{3.9}$$

The proof is complete. □

The proof of Theorem 1.2. Step 1: Fix $x_\mu \in \mathbb{R}^n$. Let

$$P_{x_\mu} = i(D_t + y\cdot D_x - \nabla_x V(x_\mu)\cdot D_y) \quad \text{and} \quad P_{X_\mu} = i(D_t + y\cdot\xi - \nabla_x V(x_\mu)\cdot D_y),$$

with $X_\mu = (x_\mu, \xi)$. Then,

$$\widehat{P_{x_\mu}u} = P_{X_\mu}\hat{u},$$

where $\hat{u}$ is the Fourier transform of u in x variable. Then from Proposition 3.1, we get

$$\langle \nabla_x V(x_\mu)\rangle^{2r/(1+r-s)}\|\hat{u}\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})} \lesssim \|\langle y\rangle^r\hat{u}\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})} + \|\langle y\rangle^s P_{X_\mu}\hat{u}\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})}. \tag{3.10}$$

Integrating both sides of (3.10) with respect to ξ, we obtain

$$\|\langle \nabla_x V(x_\mu)\rangle^{r/(1+r-s)}\hat{u}\|^2_{L^2(\mathbb{R}^{2n+1}_{t,\xi,y})} \lesssim \|\langle y\rangle^r \hat{u}\|^2_{L^2(\mathbb{R}^{2n+1}_{t,\xi,y})} + \|\langle y\rangle^s P_{X_\mu}\hat{u}\|^2_{L^2(\mathbb{R}^{2n+1}_{t,\xi,y})}. \tag{3.11}$$

By Plancherel's formula, we have

$$\|\langle \nabla_x V(x_\mu)\rangle^{r/(1+r-s)}u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} \lesssim \|\langle y\rangle^r u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} + \|\langle y\rangle^s P_{x_\mu}u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}. \tag{3.12}$$

Step 2: In this step, we denote $h(x) = \langle \nabla_x V(x)\rangle = (1+|\nabla_x V(x)|^2)^{1/2}$ for convenience. Since $V(x)$ satisfies (1.4), we consider the following metric,

$$g_x = h^\gamma(x)|dx|^2, \quad x \in \mathbb{R}^n.$$

We can know that g_x is slowly varying (one can refer to [8] and [9]). Thus we can introduce some partitions of unity related to the metric g_x. Precisely, we can find a constant $R > 0$ and a sequence $x_\mu \in \mathbb{R}^n$, $\mu \geq 1$, such that the union of the balls

$$B_{\mu,R} = \{x \in \mathbb{R}^n;\ g_{x_\mu}(x - x_\mu) < R^2\}$$

covers the whole space $\mathbb{R}^n$. Moreover there exists a positive integer N_R, depending only on R, such that the intersection of more than N_R balls is always empty. What's more, we can choose a family of nonnegative functions $\{\varphi_\mu\}_{\mu\geq 1}$ in $S(1, g_x)$ such that

$$\operatorname{Supp}\varphi_\mu \subset B_{\mu,R}, \quad \sum_{\mu\geq 1}\varphi_\mu^2 = 1, \quad \sup_\mu |\varphi_\mu|_{k,S(1,g_x)} \leq C_k,\ k \geq 0, \tag{3.13}$$

and

$$\sup_{\mu\geq 1} |\nabla_x \varphi_\mu(x)| \lesssim h^{\frac{\gamma}{2}}(x). \tag{3.14}$$

Also, we can obtain that

$$\forall x, y \in \operatorname{Supp}\varphi_\mu, \quad C^{-1}h(x) \leq h(y) \leq Ch(x). \tag{3.15}$$

Combining (3.12) and (3.15), we deduce

$$\begin{aligned}
&\|\langle \nabla_x V(x)\rangle^{\frac{r}{1+r-s}}u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} = \sum_{\mu\geq 1}\|\varphi_\mu h^{\frac{r}{1+r-s}}(x)u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} \lesssim \sum_{\mu\geq 1}\|\varphi_\mu h^{\frac{r}{1+r-s}}(x_\mu)u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} \\
&\lesssim \sum_{\mu\geq 1}\|\varphi_\mu\langle y\rangle^r u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} + \sum_{\mu\geq 1}\|\langle y\rangle^s P_{x_\mu}(\varphi_\mu u)\|^2_{L^2(R^{2n}_{x,y})} \\
&= \|\langle y\rangle^r u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} + \sum_{\mu\geq 1}\|\varphi_\mu\langle y\rangle^s P_{x_\mu}u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} + \sum_{\mu\geq 1}\|\langle y\rangle^s[P_{x_\mu}, \varphi_\mu]u\|^2_{L^2(\mathbb{R}^{2n}_{x,y})} \\
&= \|\langle y\rangle^r u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} + \sum_{\mu\geq 1}\|\varphi_\mu\langle y\rangle^s Pu\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} + \sum_{\mu\geq 1}\|\varphi_\mu\langle y\rangle^s(P_{x_\mu} - P)u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} + I_4 \\
&= \|\langle y\rangle^r u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} + \|\langle y\rangle^s f\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} + I_5 + I_4,
\end{aligned} \tag{3.16}$$

where

$$I_4 = \sum_{\mu \ge 1} \| \langle y \rangle^s [P_{x_\mu}, \varphi_\mu] u \|^2_{L^2(\mathbb{R}^{2n}_{x,y})}, \quad I_5 = \sum_{\mu \ge 1} \| \varphi_\mu \langle y \rangle^s (P_{x_\mu} - P) u \|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}. \tag{3.17}$$

Now, we estimate I_4.

$$\begin{aligned} I_4 &= \sum_{\mu \ge 1} \| \langle y \rangle^s y \cdot (\partial_x \varphi_\mu) u \|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} = \sum_{\mu \ge 1} \sum_{\nu \ge 1} \| \varphi_\nu \langle y \rangle^s y \cdot (\partial_x \varphi_\mu) u \|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} \\ &= \sum_{\mu \ge 1} \sum_{\nu \in B_\mu} \| \varphi_\nu \langle y \rangle^s y \cdot (\partial_x \varphi_\mu) u \|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} \le N_R \sum_{\mu \ge 1} \| \varphi_\mu \langle y \rangle^s y \cdot (\partial_x \varphi_\mu) u \|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} \\ &= N_R I_6, \end{aligned} \tag{3.18}$$

where

$$I_6 = \sum_{\mu \ge 1} \| \varphi_\mu \langle y \rangle^s y \cdot (\partial_x \varphi_\mu) u \|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}, \qquad B_\nu = \{\nu \ge 1; \operatorname{supp} \varphi_\nu \cap \operatorname{supp} \varphi_\mu \neq \emptyset\}.$$

B_ν is a finite set and has at most N_R elements. Using the fact (3.14), for all $\varepsilon > 0$ we have

$$\begin{aligned} I_6 &\le C \sum_{\mu \ge 1} \| \langle y \rangle^{s+1} h^{\frac{\gamma}{2}}(x) \varphi_\mu u \|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} = C \| \langle y \rangle^{s+1} h^{\frac{\gamma}{2}}(x) u \|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} \\ &\le \varepsilon \| h^{\frac{\gamma}{2} \frac{r}{r-s-1}}(x) u \|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} + C_\varepsilon \| \langle y \rangle^r u \|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} \quad \text{(by Young's inequality and } r > s+1) \\ &\le \varepsilon \| h^{\frac{r}{1+r-s}}(x) u \|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} + C_\varepsilon \| \langle y \rangle^r u \|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} \quad \left(\text{since } \gamma \le \frac{2(r-s-1)}{1+r-s}\right). \end{aligned} \tag{3.19}$$

Then we gain

$$I_4 \le \varepsilon \| \langle \nabla_x V(x) \rangle^{\frac{r}{1+r-s}} u \|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} + C_\varepsilon \| \langle y \rangle^r u \|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}. \tag{3.20}$$

Next, we calculate the term I_5.

$$I_5 = \sum_{\mu \ge 1} \| \varphi_\mu \langle y \rangle^s \big((\nabla_x V(x) - \partial_x V(x_\mu)) \cdot D_y \big) u \|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}. \tag{3.21}$$

Since $V(x)$ satisfies (1.4), by (3.13) and (3.14), we obtain

$$\begin{aligned} \sum_{\mu \ge 1} \varphi_\mu^2 |\nabla_x V(x) - \nabla_x V(x_\mu)|^2 &= \sum_{\mu \ge 1} \varphi_\mu^2 |\partial_x^2 V(x + \theta x_\mu)|^2 |x - x_\mu|^2 \quad (\theta \in (0,1)) \\ &\lesssim \sum_{\mu \ge 1} \varphi_\mu^2 \langle \nabla_x V(x + \theta x_\mu) \rangle^{2\gamma} \langle \nabla_x V(x) \rangle^{-\gamma} \\ &\lesssim \langle \nabla_x V(x) \rangle^{\gamma}. \end{aligned} \tag{3.22}$$

Combining (3.21), (3.22) and $h(x) = \langle \nabla_x V(x) \rangle$, we know

$$I_5 \lesssim \| \langle y \rangle^s h^{\frac{\gamma}{2}}(x) D_y u \|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}. \tag{3.23}$$

On the other hand,

$$\begin{aligned} \| \langle y \rangle^s h^{\frac{\gamma}{2}}(x) D_y u \|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} &\le \| \langle y \rangle^s h^{\frac{\gamma}{2}}(x) D_y \chi^w u \|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} \\ &\quad + \| \langle y \rangle^s h^{\frac{\gamma}{2}}(x) D_y (1-\chi)^w u \|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}, \end{aligned} \tag{3.24}$$

where the symbol of χ^w is

$$\chi(y,\eta) = \chi\Big(\frac{\langle y \rangle^s \langle \eta \rangle}{\varepsilon h^{\frac{r}{1+r-s} - \frac{\gamma}{2}}(x)}\Big), \quad \forall \varepsilon > 0,$$

and $\chi \in C_0^\infty(\mathbb{R}; [0,1])$ such that $\chi = 1$ in $[-1,1]$ and $\operatorname{supp} \chi \subset [-2,2]$.

Since the symbol of $\langle y \rangle^s h^{\frac{\gamma}{2}}(x) D_y \chi^w$ belongs to $S(\varepsilon h^{r/(1+r-s)}(x), |dy|^2 + |d\eta|^2)$ uniformly with respect to x, then we obtain

$$\| \langle y \rangle^s h^{\frac{\gamma}{2}}(x) D_y \chi^w u \|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} \le \varepsilon \| \langle \nabla_x V(x) \rangle^{\frac{r}{1+r-s}} u \|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}. \tag{3.25}$$

Similarly, we know that

$$\| \langle y \rangle^s h^{\frac{\gamma}{2}}(x) D_y (1-\chi)^w u \|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} \le \| \langle y \rangle^{\frac{2sr}{2r-(1+r-s)\gamma}} \langle D_y \rangle^{\frac{2r}{2r-(1+r-s)\gamma}} u \|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}.$$

On the other hand,

$$\begin{aligned} &\| \langle y \rangle^{\frac{2sr}{2r-(1+r-s)\gamma}} \langle D_y \rangle^{\frac{2r}{2r-(1+r-s)\gamma}} u \|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} \\ &\le \| \chi\Big(\frac{\langle \eta \rangle^{2r/(2r-(1+r-s)\gamma)}}{\langle y \rangle^{2r/(2r-(1+r-s)\gamma)}}\Big)^w \langle y \rangle^{\frac{2sr}{2r-(1+r-s)\gamma}} \langle D_y \rangle^{\frac{2r}{2r-(1+r-s)\gamma}} u \|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} \\ &\quad + \| \Big(1 - \chi\Big(\frac{\langle \eta \rangle^{2r/(2r-(1+r-s)\gamma)}}{\langle y \rangle^{2r/(2r-(1+r-s)\gamma)}}\Big)\Big)^w \langle y \rangle^{\frac{2sr}{2r-(1+r-s)\gamma}} \langle D_y \rangle^{\frac{2r}{2r-(1+r-s)\gamma}} u \|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} \\ &\le \| \langle y \rangle^{\frac{2r(s+1)}{2r-(1+r-s)\gamma}} u \|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} + \| \langle D_y \rangle^{\frac{2r(s+1)}{2r-(1+r-s)\gamma}} u \|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} \\ &\lesssim \| \langle y \rangle^r u \|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} + \| \langle D_y \rangle^p u \|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}. \end{aligned} \tag{3.26}$$

We use the fact that $\gamma \le \frac{2r}{1+r-s} \min\{1 - \frac{s+1}{r}, 1 - \frac{s+1}{p}\}$ in the last inequality.

Therefore, combining (3.23), (3.24), (3.25) and (3.26), we get

$$I_5 \lesssim \| \langle y \rangle^r u \|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} + \| \langle D_y \rangle^p u \|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} + \varepsilon \| \langle \nabla_x V(x) \rangle^{\frac{r}{1+r-s}} u \|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}. \tag{3.27}$$

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Then, from (3.16), (3.20) and (3.27), we can see

$$\|\langle\nabla_x V(x)\rangle^{\frac{r}{1+r-s}} u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} \lesssim \|\langle y\rangle^s f\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} + \|\langle y\rangle^r u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} + \|\langle D_y\rangle^p u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}. \qquad \square \tag{3.28}$$

Thus, we complete the proof of Theorem 1.2. Now, we consider the regularity in variable x. First, we have the following lemma.

Lemma 3.1. *For fixed $p, q \in \mathbb{R}^+$, assume that $u, g \in L^2(\mathbb{R}^{2n+1})$ satisfy the equation*

$$\partial_t u + i(y\cdot\xi - \nabla_x V(x)\cdot D_y)u = g. \tag{3.29}$$

If $\langle D_y\rangle^p u \in L^2(\mathbb{R}^{2n+1})$, $\langle D_y\rangle^q g \in L^2(\mathbb{R}^{2n+1})$, $q \neq 1+p$, then

$$\langle\xi\rangle^{\frac{2p}{1+p-q}}\|u\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})} \lesssim \|\langle D_y\rangle^p u\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})} + \|\langle D_y\rangle^q g\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})}. \tag{3.30}$$

Proof. For $q > 1+p$, (3.30) is obvious.

For $q < 1+p$, we define the symbols Q_X and g_3 by

$$Q_X(y,\eta) = D_t + y\cdot\xi - \nabla_x V(x)\cdot\eta, \quad g_3(y,\eta) = \frac{\xi\cdot\eta}{\langle\xi\rangle^{2(1-q)/(1+p-q)}}\chi\Big(\frac{\langle\eta\rangle^{p+1-q}}{\langle\xi\rangle}\Big), \tag{3.31}$$

where $\chi \in C_0^\infty(\mathbb{R};[0,1])$ such that $\chi = 1$ in $[-1,1]$ and $\operatorname{supp}\chi \subset [-2,2]$. Thus, $Q_X^{Wick} = Q_X^w = Q_X$. It is obvious that

$$\begin{aligned} Re(g, g_3^{Wick}u)_{L^2(\mathbb{R}^{n+1}_{t,y})} &= Re(iQ_X u, g_3^{Wick}u)_{L^2(\mathbb{R}^{n+1}_{t,y})} \\ &= \frac{1}{4\pi}(\{g_3, D_t + y\cdot\xi - \nabla_x V(x)\cdot\eta\}^{Wick}u, u)_{L^2(\mathbb{R}^{n+1}_{t,y})}. \end{aligned}$$

Similar to (2.4), we can also gain

$$\begin{aligned} &\{g_3, D_t + y\cdot\xi - \nabla_x V(x)\cdot\eta\} \\ &= \frac{|\xi|^2}{\langle\xi\rangle^{\frac{2(1-q)}{1+p-q}}}\chi\Big(\frac{\langle\eta\rangle^{p+1-q}}{\langle\xi\rangle}\Big) + \frac{\xi\cdot\eta}{\langle\xi\rangle^{\frac{2(1-q)}{1+p-q}}}\chi'\Big(\frac{\langle\eta\rangle^{p+1-q}}{\langle\xi\rangle}\Big)\frac{(p+1-q)\langle\eta\rangle^{p-q-1}\eta}{\langle\xi\rangle}\cdot\xi \\ &\geq \langle\xi\rangle^{2p/(1+p-q)} - \langle\eta\rangle^{2p} - \langle\xi\rangle^{-2(1-q)/(1+p-q)}. \end{aligned}$$

So, we have the following estimate,

$$\begin{aligned} &((\langle\xi\rangle^{\frac{2p}{1+p-q}})^{Wick}u, u)_{L^2(\mathbb{R}^{n+1}_{t,y})} \\ &\leq ((\langle\eta\rangle^{2p})^{Wick}u, u)_{L^2(\mathbb{R}^{n+1}_{t,y})} + ((\langle\xi\rangle^{-\frac{2(1-q)}{1+p-q}})^{Wick}u, u)_{L^2(\mathbb{R}^{n+1}_{t,y})} + |(g, g_3^{Wick}u)_{L^2(\mathbb{R}^{n+1}_{t,y})}|. \end{aligned}$$

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Next, similar to (2.4)–(2.13), we can see

$$(((\eta)^{2p})^{Wick}u,u)_{L^2(\mathbb{R}^{n+1}_{t,y})} \lesssim \|\langle D_y\rangle^p u\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})},$$

$$\begin{aligned}((\langle\xi\rangle^{-\frac{2(1-q)}{1+p-q}})^{Wick}u,u)_{L^2(\mathbb{R}^{n+1}_{t,y})} &= (\langle\xi\rangle^{-\frac{2(1-q)}{1+p-q}})u,u)_{L^2(\mathbb{R}^{n+1}_{t,y})}\\ &\lesssim \varepsilon(\langle\xi\rangle^{\frac{2p}{1+p-q}}u,u)_{L^2(\mathbb{R}^{n+1}_{t,y})} + C_\varepsilon\|u\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})},\\ |(g,g_3^{Wick}u)_{L^2(\mathbb{R}^{n+1}_{t,y})} &= |(\langle D_y\rangle^q g,\langle D_y\rangle^{-q}g_3^{Wick}u)_{L^2(\mathbb{R}^{n+1}_{t,y})}|\\ &\le C_\varepsilon\|\langle D_y\rangle^q g\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})} + \varepsilon\|\langle D_y\rangle^{-q}g_3^{Wick}u\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})}\\ &\le C_\varepsilon\|\langle D_y\rangle^q g\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})} + \varepsilon(\langle\xi\rangle^{2p/(1+p-q)}u,u)_{L^2(\mathbb{R}^{n+1}_{t,y})}.\end{aligned}$$

Thus, we get

$$\langle\xi\rangle^{2p/(1+p-q)}\|u\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})} \lesssim \|\langle D_y\rangle^p u\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})} + \|\langle D_y\rangle^q g\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})}. \qquad \square$$

The proof of Theorem 1.3. Step 1: Fix $x_\mu \in \mathbb{R}^n$. Let

$$P_{x_\mu} = i(D_t + y\cdot D_x - \partial_x V(x_\mu)\cdot D_y) \quad \text{and} \quad P_{X_\mu} = i(D_t + y\cdot\xi - \partial_x V(x_\mu)\cdot D_y),$$

with $X_\mu = (x_\mu,\xi)$. Then,

$$\widehat{P_{x_\mu}u} = P_{X_\mu}\hat{u},$$

where $\hat{u}$ is the Fourier transform of u in x variable. Then from Lemma 3.1 we know

$$\langle\xi\rangle^{\frac{2p}{1+p-q}}\|\hat{u}\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})} \lesssim \|\langle D_y\rangle^p\hat{u}\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})} + \|\langle D_y\rangle^q P_{X_\mu}\hat{u}\|^2_{L^2(\mathbb{R}^{n+1}_{t,y})}.$$

Integrating the above inequality on both sides about ξ, we can see

$$\|\langle\xi\rangle^{\frac{p}{1+p-q}}\hat{u}\|^2_{L^2(\mathbb{R}^{2n+1}_{t,\xi,y})} \lesssim \|\langle D_y\rangle^p\hat{u}\|^2_{L^2(\mathbb{R}^{2n+1}_{t,\xi,y})} + \|\langle D_y\rangle^q P_{X_\mu}\hat{u}\|^2_{L^2(\mathbb{R}^{2n+1}_{t,\xi,y})}.$$

By Plancherel's formula, we have

$$\|\langle D_x\rangle^{\frac{p}{1+p-q}}u\|_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} \lesssim \|\langle D_y\rangle^p u\|_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} + \|\langle D_y\rangle^q P_{x_\mu}u\|_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}. \tag{3.32}$$

Step 2: Here we also denote $h(x) = \langle\nabla_x V(x)\rangle = (1+|\nabla_x V(x)|^2)^{1/2}$. Taking the metric $g_x = h^\gamma(x)|dx|^2$ and the function $\varphi_\mu(x)$ satisfying (3.13), (3.14), (3.15), combining (3.32) and the Fubini theorem, we have

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$$\begin{aligned}\|\langle D_x\rangle^{\frac{p}{1+p-q}}u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} &= \sum_{\mu\ge 1}\|\varphi_\mu\langle D_x\rangle^{\frac{p}{1+p-q}}u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}\\ &\lesssim \sum_{\mu\ge 1}\|\langle D_x\rangle^{\frac{p}{1+p-q}}(\varphi_\mu u)\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}+\sum_{\mu\ge 1}\|[\varphi_\mu,\langle D_x\rangle^{\frac{p}{1+p-q}}]u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}\\ &\lesssim \sum_{\mu\ge 1}\Big[\|\langle D_y\rangle^p(\varphi_\mu u)\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}+\|\langle D_y\rangle^q P_{x_\mu}(\varphi_\mu u)\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}\Big]+I_7\\ &= \|\langle D_y\rangle^p u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}+\sum_{\mu\ge 1}\|\langle D_y\rangle^q P_{x_\mu}(\varphi_\mu u)\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}+I_7, \end{aligned}\tag{3.33}$$

where

$$I_7=\sum_{\mu\ge 1}\|[\varphi_\mu,\langle D_x\rangle^{\frac{p}{1+p-q}}]u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}.\tag{3.34}$$

Since $p\ge q+1$, we get

$$I_7\lesssim \|\langle D_x\rangle^{-\frac{1-q}{1+p-q}}u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}\le \varepsilon\|\langle D_x\rangle^{\frac{p}{1+p-q}}u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}+C_\varepsilon\|u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}.\tag{3.35}$$

Next, we know

$$\begin{aligned}&\sum_{\mu\ge 1}\|\langle D_y\rangle^q P_{x_\mu}(\varphi_\mu u)\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}\\ &\quad\lesssim \sum_{\mu\ge 1}\|\langle D_y\rangle^q P(\varphi_\mu u)\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}+\sum_{\mu\ge 1}\|\langle D_y\rangle^q(P_{x_\mu}-P)(\varphi_\mu u)\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}\\ &\quad= \sum_{\mu\ge 1}[\|\varphi_\mu\langle D_y\rangle^q Pu\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}+\|\langle D_y\rangle^q[P,\varphi_\mu]u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}]+I_8\\ &\quad= \|\langle D_y\rangle^q f\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}+I_9+I_8, \end{aligned}\tag{3.36}$$

where

$$I_8=\sum_{\mu\ge 1}\|\langle D_y\rangle^q(P_{x_\mu}-P)(\varphi_\mu u)\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})},\quad I_9=\sum_{\mu\ge 1}\|\langle D_y\rangle^q[P,\varphi_\mu]u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}.\tag{3.37}$$

Now, we calculate the term I_8.

$$\begin{aligned}I_8 &= \sum_{\mu\ge 1}\|\varphi_\mu\langle D_y\rangle^q\big((\nabla_x V(x)-\partial_x V(x_\mu))\cdot D_y\big)u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}\\ &\lesssim \|h^{\frac{\gamma}{2}}(x)\langle D_y\rangle^{q+1}u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}=\|h^{\frac{\gamma}{2}}(x)\langle\eta\rangle^{q+1}\hat{u}(t,x,\eta)\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,\eta})}, \end{aligned}\tag{3.38}$$

where $\hat{u}$ is the Fourier transform of u in y variable. By Young's inequality, we have

$$
\begin{aligned}
&\|h^{\frac{\gamma}{2}}(x)\langle\eta\rangle^{q+1}\hat{u}(t,x,\eta)\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,\eta})}\\
&\quad\lesssim \|h^{\frac{r}{1+r-s}}(x)\hat{u}(t,x,\eta)\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,\eta})}+\|\langle\eta\rangle^{\frac{2r(q+1)}{2r-\gamma(1+r-s)}}\hat{u}(t,x,\eta)\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,\eta})}\\
&\quad=\|h^{\frac{r}{1+r-s}}(x)u(t,x,y)\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}+\|\langle D_y\rangle^{\frac{2r(q+1)}{2r-\gamma(1+r-s)}}u(t,x,y)\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}.
\end{aligned}\tag{3.39}
$$

The condition (1.7) implies $\frac{2r(q+1)}{2r-\gamma(1+r-s)}\le p$. Note that $h(x)=\langle\nabla_x V(x)\rangle$, then (3.28), (3.38) and (3.39) imply

$$
I_8\lesssim\|\langle y\rangle^s f\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}+\|\langle y\rangle^r u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}+\|\langle D_y\rangle^p u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}.\tag{3.40}
$$

At last, we estimate the term I_9. Since $[P,\varphi_\mu]=y\cdot(\partial_x\varphi_\mu)$,

$$
\begin{aligned}
I_9&=\sum_{\mu\ge1}\|\langle D_y\rangle^q y\cdot(\partial_x\varphi_\mu)u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}=\sum_{\mu\ge1}\sum_{\nu\ge1}\|\varphi_\nu\langle D_y\rangle^q y\cdot(\partial_x\varphi_\mu)u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}\\
&=\sum_{\mu\ge1}\sum_{\nu\in D_\mu}\|\varphi_\nu\langle D_y\rangle^q y\cdot(\partial_x\varphi_\mu)u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}\le N_R I_{10},
\end{aligned}\tag{3.41}
$$

where

$$
I_{10}=\sum_{\mu\ge1}\|\varphi_\mu\langle D_y\rangle^q y\cdot(\partial_x\varphi_\mu)u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})},\qquad D_\nu=\{\nu\ge1;\ \mathrm{supp}\,\varphi_\nu\cap\mathrm{supp}\,\varphi_\mu\ne\emptyset\},
$$

D_ν is a finite set and has at most N_R elements.

Similar to the estimate of I_5, combining (3.14) and (3.28), we have

$$
\begin{aligned}
I_{10}&\lesssim\|h^{\frac{\gamma}{2}}(x)\langle D_y\rangle^q yu\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}\lesssim\|h^{\frac{\gamma}{2}}(x)\langle\eta\rangle^q D_\eta\hat{u}(t,x,\eta)\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,\eta})}\\
&\lesssim\varepsilon\|h^{\frac{r}{1+r-s}}(x)\hat{u}\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,\eta})}+\|\langle\eta\rangle^{\frac{2rq}{2r-\gamma(1+r-s)}}\langle D_\eta\rangle^{\frac{2r}{2r-\gamma(1+r-s)}}\hat{u}(t,x,\eta)\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,\eta})}\\
&\lesssim\varepsilon\|h^{\frac{r}{1+r-s}}(x)u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}+\|\langle\eta\rangle^{\frac{2r(q+1)}{2r-\gamma(1+r-s)}}\hat{u}(t,x,\eta)\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,\eta})}\\
&\quad+\|\langle D_\eta\rangle^{\frac{2r(q+1)}{2r-\gamma(1+r-s)}}\hat{u}(t,x,\eta)\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,\eta})}\\
&\lesssim\varepsilon\big[\|\langle y\rangle^s f\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}+\|\langle y\rangle^r u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}+\|\langle D_y\rangle^p u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}\big]\\
&\quad+\|\langle\eta\rangle^p\hat{u}(t,x,\eta)\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,\eta})}+\|\langle D_\eta\rangle^r\hat{u}(t,x,\eta)\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,\eta})}\quad(\text{by (3.28) and (1.7)})\\
&\lesssim\|\langle y\rangle^s f\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}+\|\langle D_y\rangle^p u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}+\|\langle y\rangle^r u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}.
\end{aligned}\tag{3.42}
$$

From (3.41) and (3.42), we know

$$
I_9\lesssim\|\langle y\rangle^s f\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}+\|\langle D_y\rangle^p u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}+\|\langle y\rangle^r u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}.\tag{3.43}
$$

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At last, combining (3.33), (3.36), (3.40), (3.41) and (3.43), we see

$$\|\langle D_x\rangle^{\frac{p}{1+p-q}} u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} \lesssim \|\langle D_y\rangle^p u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} + \|\langle D_y\rangle^q f\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}$$
$$+ \|\langle y\rangle^r u\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} + \|\langle y\rangle^s f\|^2_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}.$$

Therefore, we get the assertion (1.8).

Step 3: Now, we prove the regularity in t. Here, we assume that

$$0 < q < 1, \quad r \le \frac{p}{q}(s-1).$$

Let τ be the dual variable of t. Taking Fourier transform in t variable, equation (1.1) becomes

$$i\tau\hat{u} + y\cdot\nabla_x\hat{u} - \nabla_x V(x)\cdot\nabla_y\hat{u} = \hat{f}(\tau,x,y).$$

Since $q<1$, then $0<\frac{p}{1+p-q}<1$. By Young's inequality, we get

$$\begin{aligned}
&\langle y\rangle^{-\frac{p}{1+p-q}}|\tau|^{\frac{p}{1+p-q}}\\
&\lesssim \langle y\rangle^{-\frac{p}{1+p-q}}|\tau + y\cdot\nabla_x - \nabla_x V(x)\cdot\nabla_y|^{\frac{p}{1+p-q}} + \langle y\rangle^{-\frac{p}{1+p-q}}|y\cdot\nabla_x|^{\frac{p}{1+p-q}}\\
&\quad + \langle y\rangle^{-\frac{p}{1+p-q}}|\nabla_x V(x)\cdot\nabla_y|^{\frac{p}{1+p-q}}\\
&\lesssim \langle y\rangle^{-\frac{p(s+1)}{1+p-q}}(\langle y\rangle^s|\tau + y\cdot\nabla_x - \nabla_x V(x)\cdot\nabla_y|)^{\frac{p}{1+p-q}} + \langle D_x\rangle^{\frac{p}{1+p-q}} + |\nabla_x V(x)\cdot\nabla_y|^{\frac{p}{1+p-q}}\\
&\lesssim \langle y\rangle^{-\frac{p(1+s)}{1-q}} + \langle y\rangle^s|\tau + y\cdot\nabla_x - \nabla_x V(x)\cdot\nabla_y| + \langle D_x\rangle^{\frac{p}{1+p-q}} + |\nabla_x V(x)\cdot\nabla_y|^{\frac{p}{1+p-q}}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\left\| |\nabla_x V(x)\cdot\nabla_y|^{\frac{p}{1+p-q}}\hat{u}\right\|_{L^2(\mathbb{R}^{2n+1}_{\tau,x,y})}\\
&\lesssim \left\| |\nabla_x V(x)\cdot\nabla_y|^{\frac{p}{1+p-q}} u\right\|_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}\\
&\lesssim \left\| |\nabla_x V(x)\cdot\eta|^{\frac{p}{1+p-q}}\hat{u}\right\|_{L^2(\mathbb{R}^{2n+1}_{t,x,\eta})}\\
&\lesssim \|\langle\nabla_x V(x)\rangle^{\frac{r}{1+r-s}}\hat{u}\|_{L^2(\mathbb{R}^{2n+1}_{t,x,\eta})} + \|\langle\eta\rangle^{\frac{pr}{r(1-q)-p(1-s)}}\hat{u}\|_{L^2(\mathbb{R}^{2n+1}_{t,x,\eta})}\\
&\lesssim \|\langle\nabla_x V(x)\rangle^{\frac{r}{1+r-s}} u\|_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} + \|\langle D_y\rangle^{\frac{pr}{r(1-q)-p(1-s)}} u\|_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}\\
&\lesssim \|\langle y\rangle^s f\|_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} + \|\langle y\rangle^r u\|_{L^2(\mathbb{R}^{2n+1}_{t,x,y})} + \|\langle D_y\rangle^p u\|_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}.
\end{aligned}$$

The last inequality follows from (1.6) and the fact that $\frac{pr}{r(1-q)-p(1-s)} \le p$, since $r \le \frac{p}{q}(s-1)$.

Therefore, we get

$$\begin{aligned}
&\|\langle y\rangle^{-\frac{p}{1+p-q}}|D_t|^{\frac{p}{1+p-q}}u\|_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}\\
&\quad=\|\langle y\rangle^{-\frac{p}{1+p-q}}|\tau|^{\frac{p}{1+p-q}}\hat{u}\|_{L^2(\mathbb{R}^{2n+1}_{\tau,x,y})}\\
&\quad\lesssim\|\langle y\rangle^{-\frac{p(1+s)}{1-q}}\hat{u}\|_{L^2(\mathbb{R}^{2n+1}_{\tau,x,y})}+\big\|\langle y\rangle^{s}|(\tau+y\cdot\nabla_x-\nabla_x V(x)\cdot\nabla_y)\hat{u}|\big\|_{L^2(\mathbb{R}^{2n+1}_{\tau,x,y})}\\
&\qquad+\|\langle D_x\rangle^{\frac{p}{1+p-q}}\hat{u}\|_{L^2(\mathbb{R}^{2n+1}_{\tau,x,y})}+\big\||\nabla_x V(x)\cdot\nabla_y|^{\frac{p}{1+p-q}}\hat{u}\big\|_{L^2(\mathbb{R}^{2n+1}_{\tau,x,y})}\\
&\quad\lesssim\|\langle y\rangle^{s}|\hat{f}|\|_{L^2(\mathbb{R}^{2n+1}_{\tau,x,y})}+\|\langle D_x\rangle^{\frac{p}{1+p-q}}u\|_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}+\big\||\nabla_x V(x)\cdot\nabla_y|^{\frac{p}{1+p-q}}u\big\|_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}\\
&\quad\lesssim\|\langle y\rangle^{s}f\|_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}+\|\langle y\rangle^{r}u\|_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}+\|\langle D_y\rangle^{p}u\|_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}+\|\langle D_y\rangle^{q}f\|_{L^2(\mathbb{R}^{2n+1}_{t,x,y})}.
\end{aligned}$$

Thus, we obtain the assertion (1.9). □

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Appendix A

In this appendix, we recall some useful facts on symbolic calculus for readers' convenience. Let $a\in\mathcal{S}'(\mathbb{R}^{2n})$ be a temperate distribution on $\mathbb{R}^{2n}$, then the Weyl quantization is defined by

$$a^w u(y)=\int e^{2i\pi(y-z)\cdot\eta}a(\frac{y+z}{2},\eta)u(z)dzd\eta,\quad \text{for all } u\in\mathcal{S}(\mathbb{R}^n).$$

What's more, a^w is a bounded linear map from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$, and

$$\|a^w\|_{\mathcal{L}(L^2)}\le\min(2^n\|a\|_{L^1(\mathbb{R}^{2n})},\|\hat{a}\|_{L^1(\mathbb{R}^{2n})}),$$

where $\mathcal{L}(L^2)$ stands for the space of bounded linear maps from $L^2(\mathbb{R}^n)$ into itself.

Let g be an admissible metric and m an admissible weight for g (see [8] and [9]). Given a symbol $a(y,\eta)$, with or without parameters x,ξ,t,τ, we say $a\in S(m,g)$ if

$$\forall\alpha,\beta\in\mathbb{Z}^n_+,\quad\forall(y,\eta)\in\mathbb{R}^{2n},\quad|\partial_y^\alpha\partial_\eta^\beta a(y,\eta)|\le C_{\alpha,\beta}m(y,\eta),$$

with $C_{\alpha,\beta}$ a constant depending only on α,β. For $a\in S(m,g)$, we note

$$\|a\|_{k,S(m,g)}=\max_{0\le|\alpha|+|\beta|\le k}\ \sup_{(y,\eta)\in\mathbb{R}^{2n}}|m(y,\eta)^{-1}\partial_y^\alpha\partial_\eta^\beta a(y,\eta)|.$$

Then the space $S(m,g)$, with the countable family of semi-norms $(\|\cdot\|_{k,S(m,g)})_{k\in\mathbb{N}}$, is a Fréchet space. The L^2 continuity theorem in the class $S(1,|dy|^2+|d\eta|^2)$ says that, if $a\in S(1,|dy|^2+|d\eta|^2)$, then

$$\|a^w u\|_{L^2} \lesssim \|u\|_{L^2}, \quad \forall u \in L^2(\mathbb{R}^n). \tag{A.1}$$

The following composition formula is essential.

Proposition A.1. *Let g be an admissible metric on $\mathbb{R}^{2n}$ and m_1, m_2 two admissible weights for g, $a_j \in S(m_j, g)$, $j = 1, 2$. Then the symbol of the composition is $a_1 \sharp a_2 \in S(m_1 m_2, g)$ and*

$$a_1 \sharp a_2 = a_1 a_2 + \frac{1}{4i\pi}\{a_1, a_2\} + r_2, \quad r_2 \in S(m_1 m_2 \lambda_g^{-2}, g), \tag{A.2}$$

where $\lambda_g = \inf_{Y \neq 0}(\frac{g^\sigma(Y)}{g(Y)})^{1/2}$, $Y \in \mathbb{R}^{2n}$, $g^\sigma = \sigma^* g \sigma$, σ is a symplectic from in $\mathbb{R}^{2n}$ given by $\sigma((x, \xi), (y, \eta)) = \xi \cdot y - \eta \cdot x$ and $\{\cdot, \cdot\}$ is the Poisson bracket.

In particular we have the following corollary.

Corollary A.1. *For $a_j \in S^{m_j}_{1,0}$, $j = 1, 2$, we have $a_1^w a_2^w = (a_1 \sharp a_2)^w$ and*

$$\begin{aligned} a_1 \sharp a_2 &= a_1 a_2 + \frac{1}{4i\pi}\{a_1, a_2\} \mod S^{m_1+m_2-2}_{1,0}, \\ a_1 \sharp a_2 - a_2 \sharp a_1 &= \frac{1}{2i\pi}\{a_1, a_2\} \mod S^{m_1+m_2-2}_{1,0}. \end{aligned} \tag{A.3}$$

Here $S^m_{1,0} = \{a \in C^\infty(\mathbb{R}^{2n}) \mid |\partial_y^\alpha \partial_\eta^\beta a(y, \eta)| \leq C_{\alpha,\beta} \langle \eta \rangle^{m-|\beta|}, \forall \alpha, \beta \in \mathbb{Z}^n_+, \langle \eta \rangle = (1 + |\eta|^2)^{1/2}\}$, and the Poisson bracket of the symbols a_1 and a_2 is

$$\{a_1, a_2\} = \sum_{j=1}^n \frac{\partial a_1}{\partial \eta_j}\frac{\partial a_2}{\partial y_j} - \frac{\partial a_1}{\partial y_j}\frac{\partial a_2}{\partial \eta_j}.$$

Proof. It is obvious that $S^m_{1,0} = S(\langle \eta \rangle^m, |dy|^2 + \frac{|d\eta|^2}{\langle \eta \rangle^2})$. Then applying Proposition A.1 we can get the assertion (A.3). □

Finally, we introduce some basic properties of the Wick quantization, and one can refer to [9] for more detailed discussion. We define the Wick quantization of any L^∞ symbol a as

$$a^{Wick} = W^* a W,$$

where W^* is the adjoint of W and W is an isometric map from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ given by

$$Wu(y, \eta) = 2^{n/4} \int_{\mathbb{R}^n} u(z) e^{-\pi|z-y|^2} e^{2i\pi(z-y/2)\cdot\eta} dz.$$

The main property of the Wick quantization is its positivity, i.e.,

$$a(Y) \geq 0 \quad \text{for all } Y \in \mathbb{R}^{2n} \quad \text{implies} \quad a^{Wick} \geq 0, \tag{A.4}$$

which implies $(a^{Wick} u, u) \geq 0$ for all $u \in \mathcal{S}(\mathbb{R}^n)$.

The composition formula for Wick quantization is

$$a^{Wick} b^{Wick} = [ab - \frac{1}{4\pi} a' \cdot b' + \frac{1}{4i\pi}\{a, b\}]^{Wick} + r,$$

where r is a bounded operator in $L^2(\mathbb{R}^{2n})$, when $a \in L^\infty(\mathbb{R}^{2n})$ and b is a smooth symbol whose derivatives of order ≥ 2 are bounded on $\mathbb{R}^{2n}$.

What's more, the Wick and Weyl quantization of a symbol a can be linked by the following identities

$$a^{Wick} = a^w + r^w, \tag{A.5}$$

with

$$r(Y) = \int_0^1 \int_{\mathbb{R}^{2n}} (1-\theta) a''(Y + \theta Z) Z^2 e^{-2\pi |Z|^2} 2^n dZ d\theta.$$

If the symbol of a^{Wick} is a polynomial in y, η of order lower than two, then $a^{Wick} = a^w$. Thus, given a symbol in (2.1) as $Q(y, \eta) = D_t + y \cdot \xi$ with D_t, ξ as parameters, we know $Q^{Wick} = Q^w$. Using the definition of the Weyl quantization, we calculate directly that $Q^w = Q$.

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