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Vanishing viscosity limit of Navier–Stokes Equations in Gevrey class

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In this paper, we consider the inviscid limit for the periodic solutions to Navier–Stokes equation in the framework of Gevrey class. It is shown that the lifespan for the solutions to Navier–Stokes equation is independent of viscosity, and that the solutions of the Navier–Stokes equation converge to that of Euler equation in Gevrey class as the viscosity tends to zero. Moreover, the convergence rate in Gevrey class is presented. Copyright © 2017 John Wiley & Sons, Ltd.

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1. Introduction

The Navier–Stokes equations for incompressible viscous flow in $\mathbb{T}^3 = (-\pi, \pi)^3$ read

$$\begin{cases} \frac{\partial u^\nu}{\partial t} - \nu\Delta u^\nu + (u^\nu \cdot \nabla)u^\nu + \nabla p^\nu &= 0, \\ \nabla \cdot u^\nu &= 0, \\ u^\nu|_{t=0} &= a, \end{cases} \tag{1.1}$$

where $u^\nu(t,x) = (u_1^\nu, u_2^\nu, u_3^\nu)(t,x)$ is the unknown velocity vector function at point $x \in \mathbb{T}^3$ and time t, $p^\nu(t,x)$ is the unknown scalar pressure function, $\nu > 0$ is the kinematic viscosity, $a(x) = (a_1, a_2, a_3)(x)$ is the given initial data.

If the viscosity $\nu = 0$, the Eq. (1.1) becomes the Euler equation for ideal flow with the same given initial data a,

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p &= 0, \\ \nabla \cdot u &= 0, \\ u|_{t=0} &= a, \end{cases} \tag{1.2}$$

where we denote the unknown vector velocity function to be $u(t,x) = (u_1, u_2, u_3)(t,x)$ and the unknown scalar pressure function to be $p(t,x)$.

The existence and uniqueness of solutions to (1.1) and (1.2) in Sobolev space $H^r(\mathbb{R}^3)$ for $r > 3/2 + 1$, on a maximal time interval $[0, T_*)$ is classical in [1–3]. There are abundant studies on the analyticities of solutions to (1.1) and (1.2) in various methods, for reference in [4–8]. The Gevrey regularity of solutions to Navier–Stokes equations was started by Foias and Temam in their work [9], in which the authors developed a way to prove the Gevrey class regularity by characterizing the decay of their Fourier coefficients. And later, [10–14] developed this method to study the Gevrey class regularity of Euler equations in various conditions.

The subject of inviscid limits of solutions to Navier–Stokes equations has a long history, and there is a vast literature on it, investigating this problem in various functional settings, cf. [15,16] and references therein. Briefly, convergence of smooth solutions in $\mathbb{R}^n$ or torus is well developed (cf. [2, 17] for instance). Much less is known about convergence in a domain with boundaries. In fact, the vanishing viscosity limit for the incompressible Navier–Stokes equations, in the case where there exist physical boundaries, is still a challenging problem because of the appearance of the Prandtl boundary layer which is caused by the classical no-slip boundary condition. So far, the rigorous verification of the Prandtl boundary layer theory was achieved only for some specific settings, cf. [18–24] for instance, not

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to mention the convergence to Prandtl's equation and Euler equations. Several partial results on the inviscid limits, in the case of half-space, were given in [8] by imposing analyticity on the initial data, and in [25] for vorticity admitting compact support which is away from the boundary.

On the other hand, the Prandtl boundary layer equation is ill-posed in Sobolev space for many case ([19, 26, 27]), while the Sobolev space is the suitable function space for the energy theory of fluid mechanic. Because the verification of the Prandtl boundary layer theory meets the major obstacle in the setting of the Sobolev space, it will be interesting to expect the vanishing viscosity limit for the incompressible Navier–Stokes equations in the setting of Gevery space as sub-space of Sobolev space, see a series of works in this direction [20, 22, 28]. In fact, Gevrey space is an intermediate space between the space of analytic functions and the Sobolev space. On one hand, Gevrey functions enjoy similar properties as analytic functions, and on the other hand, there are nontrivial Gevrey functions having compact support, which is different from analytic functions. As a preliminary attempt, in this work, we study the vanishing viscosity limit of the solution of Navier–Stokes equation to the solution of Euler equation in Gevrey space. Here, we will concentrate on the torus, we hope this may give insights on the case when the domain has boundaries, which is a much more challenging problem.

We introduce the functions spaces as follows. We usually suppress the vector symbol for functions when no ambiguity arise. Let $\mathcal{L}^2(\mathbb{T}^3)$ be the vector function space

$$\mathcal{L}^2(\mathbb{T}^3) = \Big\{ u = \sum_{k\in\mathbb{Z}^3} \hat{u}_j e^{ik\cdot x};\ \ \hat{u}_k = \overline{\hat{u}_{-k}},\ \ \hat{u}_0 = 0,\ \ j\cdot\hat{u}_j = 0,$$
$$\|u\|_{L^2}^2 = \sum_{k\in\mathbb{Z}^3} \left|\hat{u}_k\right|^2 < \infty \Big\},$$

where $\hat{u}_k$ is the k th order Fourier coefficient of u, $i = \sqrt{-1}$. The condition $j\cdot\hat{u}_j = 0$ means $\nabla\cdot u = 0$ in the weak sense, so it is the standard L^2 space with the divergence free condition. Let $\mathcal{H}^r(\mathbb{T}^3)$ be the vector periodic Sobolev space : for $r \geq 1$,

$$\mathcal{H}^r(\mathbb{T}^3) = \Big\{ u = \sum_{k\in\mathbb{Z}^3} \hat{u}_j e^{ik\cdot x};\ \ \hat{u}_k = \overline{\hat{u}_{-k}},\ \ \hat{u}_0 = 0,\ j\cdot\hat{u}_j = 0$$
$$\|u\|_{H^r}^2 = \sum_{k\in\mathbb{Z}^3} (1+|k|^2)^r \left|\hat{u}_k\right|^2 < \infty \Big\}.$$

Here, the condition $j\cdot\hat{u}_j = 0$ means $\nabla\cdot u = 0$, so it is the standard Sobolev space H^r with the divergence free condition. Denote $(\cdot,\cdot)$ the L^2 inner product of two vector functions. Let us define the fractional differential operator $\Lambda = (-\Delta)^{1/2}$ and the exponential operator $e^{\tau\Lambda^{1/s}}$ as follows,

$$\Lambda u = \sum_{j\in\mathbb{Z}^3} |j|\,\hat{u}_j e^{ij\cdot x},\ \ e^{\tau\Lambda^{1/s}} u = \sum_{j\in\mathbb{Z}^3} e^{\tau|j|^{1/s}}\hat{u}_j e^{ij\cdot x}.$$

The vector Gevrey space $\mathcal{G}^s_{r,\tau}$ for $s \geq 1, \tau > 0, r \in \mathbb{R}$ is

$$\mathcal{G}^s_{r,\tau}(\mathbb{T}^3) = \left\{ u \in \mathcal{H}^r(\mathbb{T}^3);\quad \|u\|^2_{\mathcal{G}^s_{r,\tau}} = \sum_{j\in\mathbb{Z}^3} |j|^{2r} e^{2\tau|j|^{1/s}} \left|\hat{u}_j\right|^2 < \infty \right\},$$

where the condition $j\cdot\hat{u}_j = 0$ means $\nabla\cdot u = 0$, so it is sub-space of the Sobolev space $\mathcal{H}^r(\mathbb{T}^3)$.

The following theorem is the main result of this paper.

Theorem 1.1

Let $r > \frac{9}{2}, \tau_0 > 0, s \geq 1$. Assume that the initial data $a \in \mathcal{G}^s_{r,\tau_0}(\mathbb{T}^3)$, then there exists $\nu_0 > 0$ and $T > 0$, $\tau(t) > 0$ is a decreasing function such that, for any $0 < \nu \leq \nu_0$, the Navier–Stokes equation (1.1) admits the solutions

$$u^\nu \in L^\infty([0,T]; \mathcal{G}^s_{r,\tau(\cdot)}(\mathbb{T}^3));\ \ p^\nu \in L^\infty([0,T]; \mathcal{G}^s_{r+1,\tau(\cdot)}(\mathbb{T}^3)),$$

and the Euler equation (1.2) admits the solution

$$u \in L^\infty([0,T]; \mathcal{G}^s_{r,\tau(\cdot)}(\mathbb{T}^3));\ \ p \in L^\infty([0,T]; \mathcal{G}^s_{r+1,\tau(\cdot)}(\mathbb{T}^3)),$$

Furthermore, we have the following convergence estimates : for any $0 < t \leq T$

$$\left\|u^\nu(t,\cdot) - u(t,\cdot)\right\|_{\mathcal{G}^s_{r-1,\tau(t)}} \leq C\sqrt{\nu},\quad \left\|p^\nu(t,\cdot) - p(t,\cdot)\right\|_{\mathcal{G}^s_{r,\tau(t)}} \leq C\sqrt{\nu}, \tag{1.3}$$

where C is a constant depending on r, s, a and T.

Remark 1.1
The uniform lifespan is $0 < T < T_*$ where T_* is the maximal lifespan of $\mathcal{H}^r$ solutions. The uniform (with respect to ν) Gevrey radius $\tau(t)$ of the solution is

$$\tau(t) = \frac{1}{e^{C_1 t}\frac{1}{\tau_0} + \frac{C_2}{C_1}(e^{C_1 t} - 1)} \tag{1.4}$$

where C_1, C_2 are constants depending on r, s, a, T.

Remark 1.2
Comparison with the known works about Gevery regularity of Navier–Stokes equations and Euler equations [4, 5, 9, 11–13], the additional difficulties of this work is the uniform estimate of Gevery norm with respect to viscosity coefficients, and the estimate (1.3) with limit rates $\sqrt{\nu}$.

The paper is organized as follows. In Section 2, we will give the known results and preliminary lemmas. Section 3 consists of a priori estimate and the existence of the solutions in Gevrey space. The convergence in Gevrey space will be given in Section 4.

2. Preliminary lemmas

We first recall the following classical result of Kato in [2].

Theorem 2.1
Let $a \in \mathcal{H}^m(\mathbb{T}^3)$ for $m \geq 3$, then the following holds.

(1) There exists $T > 0$ depending on $\|a\|_{H^m}$ but not on ν, such that (1.1) has a unique solution

$$u^\nu \in C([0, T], \mathcal{H}^m(\mathbb{T}^3)) .$$

Furthermore, $\{u^\nu\}$ is bounded in $C([0, T], \mathcal{H}^m(\mathbb{T}^3))$ for all $\nu > 0$.

(2) For each $t \in [0, T]$, $u(t) = \lim_{\nu\to 0} u^\nu(t)$ exists strongly in $\mathcal{H}^{m-1}(\mathbb{T}^3)$ and weakly in $\mathcal{H}^m(\mathbb{T}^3)$, uniformly in t. u is the unique solution to (1.2) satisfying

$$u \in C([0, T], \mathcal{H}^m(\mathbb{T}^3)).$$

Remark 2.1
The time T in Theorem 2.1 is actually depending on m and $\|a\|_{H^m}$, specifically

$$T < \frac{1}{C_m \|a\|_{H^m}},$$

where C_m is a constant depending on m. In fact, the constant C_m was created by using the Leibniz formula and Sobolev embedding inequality when estimating the nonlinear term. So, if the initial data $a \in \mathcal{G}^s_{r,\tau_0}(\mathbb{T}^3)$, then we have $a \in H^m_\sigma$, $\forall m$, because there exists a constant $C_{m,\tau_0,s}$ such that $\|a\|_{H^m} \leq C_{m,\tau_0,s} \|a\|_{\mathcal{G}^s_{r,\tau_0}}$. But we can not directly obtain a uniform bound for $C_m \|a\|_{H^m}$ by the Gevrey norm of $\|a\|_{\mathcal{G}^s_{r,\tau_0}}$ when m is very large. Then, we can not say that, if m goes to infinity, $\frac{1}{C_m\|a\|_{H^m}}$ has a positive lower bound. In this paper, we will pay many attention to the uniform lifespan $T > 0$ that depends on r, $\|a\|_{H^r}$.

Remark 2.2
Compared with Theorem 2.1, the additional difficulty arises on the estimate of the convecting term in Gevrey class setting. We need to use the decaying property of the radius of Gevrey class regularity to cancel the growth of the convecting term.

We will use the following inequality, for any $j, k \in \mathbb{Z}^3\backslash\{0\}$, we have

$$|k - j| \leq 2\,|j|\,|k|\,.$$

The proof is a simple result of triangle inequality which we omit the details here. And we will give two Lemmas which will be used in the proof of Theorem 1.1.

Lemma 2.2
Given two real numbers $\xi, \eta \geq 1$ and $s \geq 1$, then the following inequality holds

$$\left|\xi^{\frac{1}{s}} - \eta^{\frac{1}{s}}\right| \leq C\frac{|\xi - \eta|}{|\xi|^{1-\frac{1}{s}} + |\eta|^{1-\frac{1}{s}}} \tag{2.1}$$

where C is a positive constant depending only on s.

Proof
The case for $s = 1$ is trivial. Let us consider the case for $s > 1$. Without loss of generality, we may assume $\xi > \eta$. Then, (2.1) is equivalent with

$$\frac{(\xi^{\frac{1}{s}} - \eta^{\frac{1}{s}})(\xi^{1-\frac{1}{s}} + \eta^{1-\frac{1}{s}})}{\xi - \eta} \leq C.$$

Then, it suffices to show that

$$\left|\frac{(\frac{\eta}{\xi})^{1-\frac{1}{s}}-(\frac{\eta}{\xi})^{\frac{1}{s}}}{1-\frac{\eta}{\xi}}\right|\le C.$$

By Theorem 42 in [29], it can be obtained for fixed $s>1$

$$\left|\frac{(\frac{\eta}{\xi})^{1-\frac{1}{s}}-(\frac{\eta}{\xi})^{\frac{1}{s}}}{1-\frac{\eta}{\xi}}\right|\le \max\left(1-\frac{2}{s},\frac{2}{s}-1\right)\le C$$

Then, the lemma 2.2 is proved. □

With the use of Lemma 2.2, we have the following estimate about the nonlinear term.

Lemma 2.3
Let $r>\frac{9}{2}, s\ge 1$ and $\tau>0$ is a constant. Then, for any $v\in\mathcal{G}^s_{r+1,\tau}(\mathbb{T}^3)$, the following estimate holds,

$$\begin{aligned}\left|\left(\Lambda^r e^{\tau\Lambda^{1/s}}(v\cdot\nabla v),\ \Lambda^r e^{\tau\Lambda^{1/s}}v\right)\right| &\le C\,\|v\|_{H^r}\,\|v\|^2_{\mathcal{G}^s_{r,\tau}}+C\,\|v\|^2_{H^r}\,\|v\|_{\mathcal{G}^s_{r,\tau}}\\ &\quad+\left[C\tau\,\|u\|_{H^r}+C\tau^2(\|u\|_{H^r}+\|u\|_{\mathcal{G}^s_{r,\tau}})\right]\|u\|^2_{\mathcal{G}^s_{r+\frac{1}{2s},\tau}},\end{aligned}\tag{2.2}$$

where C is a constant depending only on r and s.

Proof
By the definition of the vector function space $\mathcal{G}^s_{r+1,\tau}(\mathbb{T}^3)$, we have $v=\sum_{j\in\mathbb{Z}^3}\hat v_j e^{ij\cdot x}$ and $\hat v_0=0$. Using Fourier series convolution property, one has

$$v\cdot\nabla v=i\sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}[\hat v_j\cdot(k-j)]\hat v_{k-j}e^{ik\cdot x}.$$

Applying the operator $\Lambda^r e^{\tau\Lambda^{1/s}}$ on $v\cdot\nabla v$, one has

$$\Lambda^r e^{\tau\Lambda^{1/s}}(v\cdot\nabla v)=i\sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}[\hat v_j\cdot(k-j)]\hat v_{k-j}\,|k|^r\,e^{\tau|k|^{1/s}}e^{ik\cdot x}.$$

And $\Lambda^r e^{\tau\Lambda^{1/s}}v=\sum_{\ell\in\mathbb{Z}^3}|\ell|^r\,e^{\tau|\ell|^{1/s}}\hat v_\ell e^{i\ell\cdot x}$. Now, we take the L^2 inner product of $\Lambda^r e^{\tau\Lambda^{1/s}}(v\cdot\nabla v)$ with $\Lambda^r e^{\tau\Lambda^{1/s}}v$ over $\mathbb{T}^3$. The orthogonality of the exponentials in L^2 implies

$$\begin{aligned}&\left(\Lambda^r e^{\tau\Lambda^{1/s}}(v\cdot\nabla v),\Lambda^r e^{\tau\Lambda^{1/s}}v\right)\\ &=i(2\pi)^3\sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}\left[\hat v_j\cdot(k-j)\right](\hat v_{k-j}\cdot\hat v_{-k})\,|k|^{2r}\,e^{2\tau|k|^{1/s}}.\end{aligned}$$

The cancelation property of the convecting term implies

$$\begin{aligned}0&=\left(v\cdot\nabla\Lambda^r e^{\tau\Lambda^{1/s}}v,\Lambda^r e^{\tau\Lambda^{1/s}}v\right)\\ &=i(2\pi)^3\sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}\left[\hat v_j\cdot(k-j)\right]|k-j|^r\,e^{\tau|k-j|^{1/s}}(\hat v_{k-j}\cdot\hat v_{-k})\,|k|^r\,e^{\tau|k|^{1/s}}.\end{aligned}$$

Then, we have

$$\begin{aligned}&\left(\Lambda^r e^{\tau\Lambda^{1/s}}(v\cdot\nabla v),\Lambda^r e^{\tau\Lambda^{1/s}}v\right)\\ &=\left(\Lambda^r e^{\tau\Lambda^{1/s}}(v\cdot\nabla v)-v\cdot\nabla\Lambda^r e^{\tau\Lambda^{1/s}}v,\Lambda^r e^{\tau\Lambda^{1/s}}v\right)\\ &=\mathcal{T}_1+\mathcal{T}_2,\end{aligned}$$

where

$$\mathcal{T}_1=i(2\pi)^3\sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}(|k|^r-|k-j|^r)e^{\tau|k-j|^{1/s}}\left[\hat v_j\cdot(k-j)\right](\hat v_{k-j}\cdot\hat v_{-k})\,|k|^r\,e^{\tau|k|^{1/s}},$$

and

$$\mathcal{T}_2=i(2\pi)^3\sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}|k|^r\,(e^{\tau|k|^{1/s}}-e^{\tau|k-j|^{1/s}})\left[\hat v_j\cdot(k-j)\right](\hat v_{k-j}\cdot\hat v_{-k})\,|k|^r\,e^{\tau|k|^{1/s}}.$$

Before we come to the estimate of $\mathcal{T}_1$ and $\mathcal{T}_2$, we recall the following mean value theorem; for $\forall\xi,\eta\in\mathbb{R}^+$, there exists a constant $0\le\theta,\theta'\le 1$ such that

$$
\begin{aligned}
\xi^r - \eta^r &= r(\xi-\eta)\left[(\theta\xi + (1-\theta)\eta)^{r-1} - \xi^{r-1}\right] + r(\xi-\eta)\eta^{r-1} \\
&= r(r-1)\theta(\xi-\eta)^2[\theta'(\theta\xi+(1-\theta)\eta) + (1-\theta')\eta]^{r-2} \\
&\quad + r(\xi-\eta)\eta^{r-1}.
\end{aligned}
$$

Then, there exists a constant C depending only on r such that

$$
\left||k|^r - |k-j|^r\right| \le C\,|j|^2\,(|j|^{r-2} + |k-j|^{r-2}) + C\,|j|\,|k-j|^{r-1}\,.
$$

From the inequality $e^{\xi} \le e + \xi^2 e^{\xi}$ that holds for all $\xi \in \mathbb{R}$, we can be bounded by the exponential $e^{\tau|k-j|^{1/s}}$ by $e + \tau^2\,|k-j|^{2/s}\,e^{\tau|k-j|^{1/s}}$. Then, $\mathcal{T}_1$ can be bounded by

$$
\begin{aligned}
&|\mathcal{T}_1| \\
&\le C\sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}(|j|^r + |j|^2\,|k-j|^{r-2})\,|\hat{v}_j|\,|k-j|\,|\hat{v}_{k-j}|\,(e + \tau^2\,|k-j|^{2/s}\,e^{\tau|k-j|^{1/s}}) \\
&\quad\times|\hat{v}_{-k}|\,|k|^r\,e^{\tau|k|^{1/s}} + C\sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}|j|\,|\hat{v}_j|\,|k-j|^r\,e^{\tau|k-j|^{1/s}}\,|\hat{v}_{k-j}|\,|\hat{v}_{-k}|\,|k|^r\,e^{\tau|k|^{1/s}} \\
&= \mathcal{T}_{11} + \mathcal{T}_{12} + \mathcal{T}_{13} + \mathcal{T}_{14} + \mathcal{T}_{15}.
\end{aligned}
$$

With application of discrete Hölder inequality and Minkowski inequality, one can obtain the following estimates. For example, we give the details for $\mathcal{T}_{11}$, and the rest can be estimated in the same way,

$$
\begin{aligned}
\mathcal{T}_{11} &= eC\sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}|j|^r\,|\hat{v}_j|\,|k-j|\,|\hat{v}_{k-j}|\,|k|^r\,e^{\tau|k|^{1/s}}\,|\hat{v}_{-k}| \\
&\le C\,\|v\|_{\mathcal{G}^s_{r,\tau}}\left[\sum_{k\in\mathbb{Z}^3}\left(\sum_{j\in\mathbb{Z}^3}|j|^r\,|\hat{v}_j|\,|k-j|\,|\hat{v}_{k-j}|\right)^2\right]^{1/2} \\
&= C\,\|v\|_{\mathcal{G}^s_{r,\tau}}\left[\sum_{k\in\mathbb{Z}^3}\left(\sum_{\ell\in\mathbb{Z}^3}|k-\ell|^r\,|\hat{v}_{k-\ell}|\,|\ell|\,|\hat{v}_\ell|\right)^2\right]^{1/2} \\
&\le C\,\|v\|_{\mathcal{G}^s_{r,\tau}}\left(\sum_{k\in\mathbb{Z}^3}|k-\ell|^{2r}\,|\hat{v}_{k-\ell}|^2\right)^{1/2}\sum_{\ell\in\mathbb{Z}^3}\frac{|\ell|}{(1+|\ell|^2)^{r/2}}(1+|\ell|^2)^{r/2}\,|\hat{v}_\ell| \\
&\le C\,\|v\|^2_{H^r}\,\|v\|_{\mathcal{G}^s_{r,\tau}}\left(\sum_{\ell\in\mathbb{Z}^3}\frac{|\ell|^2}{(1+|\ell|^2)^r}\right)^{1/2} \\
&\le C\,\|v\|^2_{H^r}\,\|v\|_{\mathcal{G}^s_{r,\tau}}\,,
\end{aligned}
$$

where C is a constant depending on r, e and for $r > 9/2$, the summation in the aforementioned $\left(\sum_{\ell\in\mathbb{Z}^3}\frac{|\ell|^2}{(1+|\ell|^2)^r}\right)^{1/2}$ is bounded by some constant depending on r. Similarly, with $\mathcal{T}_{11}$, we have

$$
\begin{aligned}
\mathcal{T}_{12} &= eC\sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}|j|^2\,|\hat{v}_j|\,|k-j|^{r-1}\,|\hat{v}_{k-j}|\,|k|^r\,e^{\tau|k|^{1/s}}\,|\hat{v}_{-k}| \\
&\le C\,\|v\|^2_{H^r}\,\|v\|_{\mathcal{G}^s_{r,\tau}}\,,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{T}_{13} &= C\tau^2\sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}|j|^r\,|\hat{v}_j|\,|k-j|^{1+2/s}\,|\hat{v}_{k-j}|\,|k|^r\,e^{\tau|k|^{1/s}}\,|\hat{v}_{-k}| \\
&\le C\tau^2\,\|v\|_{H^r}\,\|v\|^2_{\mathcal{G}^s_{r,\tau}}\,.
\end{aligned}
$$

Note that $\hat{v}_0 = 0, s \ge 1$ in the summation, and $|k-j|^{\frac{1}{2s}} \le C\,|k|^{\frac{1}{2s}}\,|j|^{\frac{1}{2s}}$, we can similarly have

$$
\begin{aligned}
\mathcal{T}_{14} &= C\tau^2\sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}|j|^2\,|\hat{v}_j|\,|k-j|^{r-1+2/s}\,e^{\tau|k-j|^{1/s}}\,|\hat{v}_{k-j}|\,|k|^r\,e^{\tau|k|^{1/s}}\,|\hat{v}_{-k}| \\
&\le C\tau^2\sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}|j|^{2+\frac{1}{2s}}\,|\hat{v}_j|\,|k-j|^{r+\frac{1}{2s}}\,e^{\tau|k-j|^{1/s}}\,|\hat{v}_{k-j}|\,|k|^{r+\frac{1}{2s}}\,e^{\tau|k|^{1/s}}\,|\hat{v}_{-k}| \\
&\le C\tau^2\,\|v\|_{H^r}\,\|v\|^2_{\mathcal{G}^s_{r+\frac{1}{2s},\tau}}\,,
\end{aligned}
$$

and

$$\begin{aligned}\mathcal{T}_{15} &= C\sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}|j|\,|\hat v_j|\,|k-j|^r\,e^{\tau|k-j|^{1/s}}\,|\hat v_{k-j}|\,|k|^r\,e^{\tau|k|^{1/s}}\,|\hat v_{-k}|\\ &\le C\,\|v\|_{H^r}\,\|v\|^2_{\mathcal{G}^s_{r,\tau}}\,.\end{aligned}$$

Noting that $\|v\|_{\mathcal{G}^s_{r,\tau}}\le\|v\|_{\mathcal{G}^s_{r+\frac{1}{2s},\tau}}$, then $\mathcal{T}_{13}\le\mathcal{T}_{14}$. Thus, we obtain

$$|\mathcal{T}_1|\le C\,\|v\|^2_{H^r}\,\|v\|_{\mathcal{G}^s_{r,\tau}}+C\,\|v\|_{H^r}\,\|v\|^2_{\mathcal{G}^s_{r,\tau}}+C\tau^2\,\|v\|_{H^r}\,\|v\|^2_{\mathcal{G}^s_{r+\frac{1}{2s},\tau}}\,.$$

As for $\mathcal{T}_2$, we have

$$\begin{aligned}\mathcal{T}_2 &= i(2\pi)^3\sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}|k|^r\,(e^{\tau|k|^{1/s}}-e^{\tau|k-j|^{1/s}})[\hat v_j\cdot(k-j)](\hat v_{k-j}\cdot\hat v_{-k})\,|k|^r\,e^{\tau|k|^{1/s}}\\ &= i(2\pi)^3\sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}|k|^r\,e^{\tau|k-j|^{1/s}}\left[e^{\tau(|k|^{1/s}-|k-j|^{1/s})}-1\right][\hat v_j\cdot(k-j)]\\ &\quad\times(\hat v_{k-j}\cdot\hat v_{-k})\,|k|^r\,e^{\tau|k|^{1/s}}.\end{aligned}$$

We note that the inequality $\left|e^{\xi}-1\right|\le|\xi|\,e^{|\xi|}$ holds for $\xi\in\mathbb{R}$. Then

$$\left|e^{\tau(|k|^{1/s}-|k-j|^{1/s})}-1\right|\le\tau\left||k|^{1/s}-|k-j|^{1/s}\right|e^{\tau\left||k|^{1/s}-|k-j|^{1/s}\right|}.$$

Because $s\ge1$, we have

$$\left||k|^{1/s}-|k-j|^{1/s}\right|\le|j|^{1/s}.$$

Then, we actually have

$$\left|e^{\tau(|k|^{1/s}-|k-j|^{1/s})}-1\right|\le\tau\left||k|^{1/s}-|k-j|^{1/s}\right|e^{\tau|j|^{1/s}}.$$

By Lemma 2.2, we have

$$\left||k|^{1/s}-|k-j|^{1/s}\right|\le C\,|j|\,\frac{1}{|k|^{1-1/s}+|k-j|^{1-1/s}}.$$

Then, $\mathcal{T}_2$ can be bounded by the inequality

$$\begin{aligned}|\mathcal{T}_2| &\le C\sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}|k|^r\,e^{\tau|k-j|^{1/s}}\left|e^{\tau(|k|^{1/s}-|k-j|^{1/s})}-1\right||k-j|\,|\hat v_j|\,|\hat v_{k-j}|\\ &\quad\times|v_{-k}|\,|k|^r\,e^{\tau|k|^{1/s}}\\ &\le C\sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}|k|^{r-\frac{1}{2s}}\,e^{\tau|k-j|^{1/s}}\,\tau\left||k|^{1/s}-|k-j|^{1/s}\right|e^{\tau|j|^{1/s}}\,|k-j|\,|\hat v_j|\,|\hat v_{k-j}|\\ &\quad\times\left|\hat v_{-k}\right||k|^{r+\frac{1}{2s}}\,e^{\tau|k|^{1/s}}\\ &\le C\tau\sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}(|j|^{r-\frac{1}{2s}}+|k-j|^{r-\frac{1}{2s}})e^{\tau|k-j|^{1/s}}\,\frac{|j|\,|k-j|}{|k|^{1-1/s}+|k-j|^{1-1/s}}\\ &\quad\times e^{\tau|j|^{1/s}}\,|\hat u_j|\,|\hat v_{k-j}|\,|\hat v_{-k}|\,|k|^{r+\frac{1}{2s}}\,e^{\tau|k|^{1/s}}\\ &\le\mathcal{T}_{21}+\mathcal{T}_{22},\end{aligned}$$

where

$$\begin{aligned}\mathcal{T}_{21} &= C\tau\sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}|j|^{r+\frac{1}{2s}}\,e^{\tau|j|^{1/s}}\,|\hat v_j|\,|k-j|\,(1+\tau\,|k-j|^{1/s}\,e^{\tau|k-j|^{1/s}})\\ &\quad\times|\hat v_{k-j}|\,|k|^{r+\frac{1}{2s}}\,e^{\tau|k|^{1/s}}\,|\hat v_{-k}|,\\ \mathcal{T}_{22} &= C\tau\sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}|j|\,(1+\tau\,|j|^{1/s}\,e^{\tau|j|^{1/s}})\,|\hat v_j|\,|k-j|^{r+\frac{1}{2s}}\,e^{\tau|k-j|^{1/s}}\\ &\quad\times|\hat v_{k-j}|\,|k|^{r+\frac{1}{2s}}\,e^{\tau|k|^{1/s}}\,|\hat v_{-k}|.\end{aligned}$$

We have used the inequality $|k|^{1-1/s}+|k-j|^{1-1/s}\ge|j|^{1-1/s}$ and $e^{\xi}\le1+\xi e^{\xi}$ for $\xi\in\mathbb{R}^+$ in the estimation of $\mathcal{T}_{21}$. With application of Hölder inequality and Minkowiski inequality, we have for $\mathcal{T}_{21}$,

$$|\mathcal{T}_{21}| \le C\tau \|v\|_{H^r} \|v\|^2_{\mathcal{G}^s_{r+\frac{1}{2s},\tau}} + C\tau^2 \|v\|_{\mathcal{G}^s_{r,\tau}} \|v\|^2_{\mathcal{G}^s_{r+\frac{1}{2s},\tau}}.$$

Symmetrically, one has a same bound for $\mathcal{T}_{22}$, then for $\mathcal{T}_2$,

$$|\mathcal{T}_{2}| \le C\tau \|v\|_{H^r} \|v\|^2_{\mathcal{G}^s_{r+\frac{1}{2s},\tau}} + C\tau^2 \|v\|_{\mathcal{G}^s_{r,\tau}} \|v\|^2_{\mathcal{G}^s_{r+\frac{1}{2s},\tau}}.$$

Then, we obtain

$$\begin{aligned}\left|\left(\Lambda^r e^{\tau\Lambda^{1/s}}(v\cdot\nabla v), \Lambda^r e^{\tau\Lambda^{1/s}} v\right)\right| &\le |\mathcal{T}_1| + |\mathcal{T}_2| \\ &\le C\|v\|^2_{H^r}\|v\|_{\mathcal{G}^s_{r,\tau}} + C\|v\|_{H^r}\|v\|^2_{\mathcal{G}^s_{r,\tau}} \\ &\quad + C\tau^2\|v\|_{\mathcal{G}^s_{r,\tau}}\|v\|^2_{\mathcal{G}^s_{r+\frac{1}{2s},\tau}} \\ &\quad + C\tau(1+\tau)\|v\|_{H^r}\|v\|^2_{\mathcal{G}^s_{r+\frac{1}{2s},\tau}},\end{aligned}$$

which finishes the proof of Lemma 2.3. □

3. Uniform existence of solutions

In this section, we will first show the existence of Gevrey class solutions u^ν to Navier–Stokes equation (1.1). And the existence of Gevrey class solution u to Euler equation (1.2) can be obtained similarly. The method of the proof are based on Galerkin approximation. Before that, we first consider the following equivalent equation for Navier–Stokes equation,

$$\begin{aligned}\frac{d}{dt}u^\nu + \nu A u^\nu + \mathbb{P}(u^\nu\cdot\nabla u^\nu) = 0,\\ u^\nu|_{t=0} = \mathbb{P}a.\end{aligned} \tag{3.1}$$

where $A = -\mathbb{P}\Delta$ is the well-known Stokes operator and $\mathbb{P}$ is the Leray projector which maps a vector function v into its divergence free part v_1, such that $v = v_1 + \nabla q$ and $\nabla\cdot v_1 = 0$, q is a scalar function and $(v_1, \nabla q) = 0$. Similarly, for Euler equation, we have the following equivalent form,

$$\begin{aligned}\frac{d}{dt}u^\nu + \mathbb{P}(u^\nu\cdot\nabla u^\nu) = 0,\\ u^\nu|_{t=0} = \mathbb{P}a.\end{aligned} \tag{3.2}$$

We then recall some properties of the Stokes operator A, which are known in [Chapter 4 in [30]].

Proposition 3.1
The Stokes operator A is symmetric and self-adjoint; moreover, the inverse of the Stokes operator, A^{-1}, is a compact operator in $\mathcal{L}^2$. The Hilbert theorem implies that there exists a sequence of positive numbers λ_j and an orthonormal basis w_j of $\mathcal{L}^2$, which satisfies

$$Aw_j = \lambda_j w_j,\ 0 < \lambda_1 < \ldots < \lambda_j \le \lambda_{j+1} \le \ldots,\ \lim_{j\to\infty}\lambda_j = \infty.$$

Moreover, in the case of $\mathbb{T}^3 = (-\pi,\pi)^3$, the sequence of eigenvector functions $w_j's$ and eigenvalues $\lambda_j's$ are the sequences of functions $w_{k,j}$ and numbers $\lambda_{k,j}$,

$$w_{k,j}(x) = \left(e_j - \frac{k_j k}{|k|^2}\right)e^{ik\cdot x},\quad \lambda_{k,j} = |k|^2,$$

where $k = (k_1,k_2,k_3)\in\mathbb{Z}^3$, $k\neq 0$, $j = 1,2,3$ and $\{e_j\}_{j=1,2,3}$ are the canonical basis in $\mathbb{R}^3$. So, we know that each w_j are not only in $\mathcal{L}^2$, but also in $\mathcal{G}^s_{r,\tau}$ for $\forall r > 0$. Now, we will show that there exists a solution to Eq. (3.1) for $a\in\mathcal{G}^s_{r,\tau}$ with $r > 9/2$, $s \ge 1$, and $\tau(t) > 0$ is a differentiable decreasing function of t. To this end, we first prove a priori estimate in the following Proposition.

Proposition 3.2
Let $r > 9/2, s \ge 1, a\in\mathcal{G}^s_{r,\tau_0}$ and $\tau(t) > 0$ is a differentiable decreasing function of t defined on $[0,T]$ with $\tau(0) = \tau_0 > 0$, where $0 < T < T_*$ and T_* is the maximal time of H^r solution to (3.1) with respect to the initial data a. Let $u^\nu(t,x)\in L^\infty([0,T];\mathcal{G}^s_{r,\tau(\cdot)}(\mathbb{T}^3))\cap L^2([0,T];\mathcal{G}^s_{r+1,\tau(\cdot)}(\mathbb{T}^3))$ be the solution to (3.1), then the following a priori estimates holds

$$\begin{aligned}&\|u^\nu(t,\cdot)\|_{\mathcal{G}^s_{r,\tau(t)}} \le G_T,\quad 0 < t \le T,\\ &\nu\int_0^t \|u^\nu(s,\cdot)\|^2_{\mathcal{G}^s_{r+1,\tau(s)}}ds \le M_T,\quad 0 < t \le T.\end{aligned}$$

With the same assumptions as earlier, let $u(t,x)\in L^\infty([0,T];\mathcal{G}^s_{r+1,\tau(\cdot)}(\mathbb{T}^3))$ be the solution to (3.2), we also have

$$\|u(t,\cdot)\|_{\mathcal{G}^s_{r,\tau(t)}} \le G_T,\quad 0 < t \le T,$$

Furthermore, the uniform radius $\tau(t)$ is given by

$$\tau(t) = \frac{1}{e^{C_1 t}\frac{1}{\tau_0} + \frac{C_2}{C_1}(e^{C_1 t} - 1)},$$

where C, C_1, C_2, G_T, M_T are constants depending on a, r, s, T.

Proof

Applying $\Lambda^r e^{\tau\Lambda^{1/s}}$ on both sides of (3.1) and taking the L^2 inner product of both sides with $\Lambda^r e^{\tau\Lambda^{1/s}} u^\nu$, one has

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|u^\nu(t,\cdot)\right\|^2_{\mathcal{G}^s_{r,\tau(t)}} + \nu\left\|u^\nu(t,\cdot)\right\|^2_{\mathcal{G}^s_{r+1,\tau(t)}} \\ &\quad = \tau'(t)\left\|u^\nu(t,\cdot)\right\|^2_{\mathcal{G}^s_{r+\frac{1}{2s},\tau}} - \left(\Lambda^r e^{\tau\Lambda^{1/s}}(u^\nu\cdot\nabla u^\nu), \Lambda^r e^{\tau\Lambda^{1/s}} u^\nu\right),\end{aligned} \tag{3.3}$$

where we use the fact that $\mathbb{P}$ commutes with $\Lambda^r e^{\tau\Lambda^{1/s}}$ and $\mathbb{P}$ is symmetric. Now, we consider the right hand side of (3.3). By (2.2) in Lemma 2.3, we have from (3.3); for convenience, we sometimes suppress the dependence of u^ν and τ in t,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|u^\nu\right\|^2_{\mathcal{G}^s_{r,\tau}} + \nu\left\|u_\nu\right\|^2_{\mathcal{G}^s_{r+1,\tau}} \le C\left\|u^\nu\right\|_{H^r}\left\|u^\nu\right\|^2_{\mathcal{G}^s_{r,\tau}} + C\left\|u^\nu\right\|^2_{H^r}\left\|u^\nu\right\|_{\mathcal{G}^s_{r,\tau}} \\ &+ (\tau' + C\tau\left\|u^\nu\right\|_{H^r} + C\tau^2\left\|u^\nu\right\|_{H^r} + C\tau^2\left\|u^\nu\right\|_{\mathcal{G}^s_{r,\tau}})\left\|u^\nu\right\|^2_{\mathcal{G}^s_{r+\frac{1}{2s},\tau}},\end{aligned} \tag{3.4}$$

where C is a constant depending on r, s. Now, if the radius of Gevrey class $\tau(t)$ is smooth and decreaseing fast enough such that the following inequality holds,

$$\tau' + C\tau\left\|u^\nu\right\|_{H^r} + C\tau^2\left\|u^\nu\right\|_{H^r} + C\tau^2\left\|u^\nu\right\|_{\mathcal{G}^s_{r,\tau}} \le 0. \tag{3.5}$$

Then, (3.4) implies

$$\frac{1}{2}\frac{d}{dt}\left\|u^\nu\right\|^2_{\mathcal{G}^s_{r,\tau}} + \nu\left\|u^\nu\right\|^2_{\mathcal{G}^s_{r+1,\tau}} \le C\left\|u^\nu\right\|_{H^r}\left\|u^\nu\right\|^2_{\mathcal{G}^s_{r,\tau}} + C\left\|u^\nu\right\|^2_{H^r}\left\|u^\nu\right\|_{\mathcal{G}^s_{r,\tau}}. \tag{3.6}$$

As $\nu > 0$, it can be obtained directly from (3.6),

$$\frac{d}{dt}\left\|u^\nu\right\|_{\mathcal{G}^s_{r,\tau}} \le C\left\|u^\nu\right\|_{H^r}\left\|u^\nu\right\|_{\mathcal{G}^s_{r,\tau}} + C\left\|u^\nu\right\|^2_{H^r}, \tag{3.7}$$

By Grownwall's inequality in (3.7), we have for $0 < t < T_*$,

$$\left\|u^\nu(t)\right\|_{\mathcal{G}^s_{r,\tau(t)}} \le \tilde{g}(t)\left(\|a\|_{\mathcal{G}^s_{r,\tau_0}} + \int_0^t C\tilde{g}(s)^{-1}\left\|u^\nu(s)\right\|^2_{H^r}ds\right) \triangleq A(t), \tag{3.8}$$

where $\tilde{g}(t) = e^{\int_0^t C\|u^\nu(s,\cdot)\|_{H^r}ds}$ and T_* is the maximal time interval of H^r solution. It has been known that T_* is independent of ν. Moreover, it follows the a priori estimate for H^r solution for Navier–Stokes equation in [31],

$$\frac{d}{dt}\left\|u^\nu(t,\cdot)\right\|^2_{H^r} \le C\left\|u^\nu(t,\cdot)\right\|_{H^r}\left\|u^\nu(t,\cdot)\right\|^2_{H^r}, \quad 0 < t < T_*, \tag{3.9}$$

where C is a constant depending on r. Then, on one side, we have

$$\left\|u^\nu(t)\right\|^2_{H^r} \le C\tilde{g}(t)\left\|a\right\|^2_{H^r}, \quad 0 < t < T_*. \tag{3.10}$$

And on the other side, let $0 < T < T_*$, then for $0 < t < T$,

$$\left\|u^\nu(t,\cdot)\right\|_{H^r} \le \frac{\|a\|_{H^r}}{1 - Ct\left\|a\right\|_{H^r}} \le \frac{\|a\|_{H^r}}{1 - CT\left\|a\right\|_{H^r}} \triangleq C_T. \tag{3.11}$$

With (3.9),(3.10), and (3.11), we have

$$\begin{aligned}\left\|u^\nu(t,\cdot)\right\|_{\mathcal{G}^s_{r,\tau(t)}} &\le A(t) \\ &= e^{\int_0^t C\|u^\nu(s)\|_{H^r}ds}\left(\|a\|_{G^s_{r,\tau_0}} + \int_0^t C\tilde{g}(s)^{-1}\left\|u^\nu(s)\right\|^2_{H^r}ds\right) \\ &\le e^{CC_T t}\left(\|a\|_{G^s_{r,\tau_0}} + C\left\|a\right\|^2_{H^r}t\right) \\ &\le e^{CC_T T}\left(\|a\|_{G^s_{r,\tau_0}} + C\left\|a\right\|^2_{H^r}T\right) \triangleq G_T, \quad 0 \le t \le T.\end{aligned} \tag{3.12}$$

In fact, a sufficient condition for (3.5) to hold is

$$\tau'(t) + C\tau(t)C_T + C\tau^2(t)C_T + C\tau^2(t)G_T = 0. \tag{3.13}$$

Then, solving the ordinary differential Eq. (3.13),

$$\tau(t) = \frac{1}{e^{CC_T t}\frac{1}{\tau_0} + \frac{CC_T + CG_T}{CC_T}(e^{CC_T t} - 1)}. \tag{3.14}$$

We can obtain (1.4) by arranging the constants in (3.14),

$$\tau(t) = \frac{1}{e^{C_1 t}\frac{1}{\tau_0} + \frac{C_2}{C_1}(e^{C_1 t} - 1)}, \tag{3.15}$$

where C_1, C_2 are constants depending on r, s, T, a. Then (3.15) proves (1.4) in Remark 1.1. Integrating (3.6) form 0 to t, we have, for $0 < t < T$,

$$\nu \int_0^t \left\| u^\nu(s,\cdot) \right\|^2_{\mathcal{G}^s_{r+1,\tau(s)}} ds \leq M_T, \;\; 0 < t < T < T_*, \tag{3.16}$$

where M_T depends on T, a, r, s, τ_0. It should be noted that all of the aforementioned estimates are independent of ν, so we let $\nu = 0$ in (3.4), and proceed exactly as earlier, then we can obtain similar results for the a priori estimate for solution u to Eq. 3.2. □

With the estimates in Proposition 3.2, we can implement the standard Faedo–Fourier–Galerkin approximation as in [32, 33] to prove the existence of such u^ν and u in the function space of Gevrey class s $\mathcal{G}^s_{r,\tau}$.

Theorem 3.1
There exists a unique solution u^ν to (3.1) such that

$$u^\nu \in L^\infty([0,T], \mathcal{G}^s_{r,\tau(\cdot)}).$$

Similarly, there exists a unique solution u to (3.2) such that

$$u \in L^\infty([0,T], \mathcal{G}^s_{r,\tau(\cdot)}).$$

Proof
The method of proof of existence is based on Galerkin approximations and the priori estimate in Proposition 3.2. For a fixed positive integer n, we will look for a sequence of functions $u^\nu_n(t,\cdot)$ with $n \in \mathbb{N}$ in the form

$$u^\nu_n(t,x) = \sum_{j=1}^n \alpha^\nu_{j,n}(t)\omega_j,$$

where $\{\omega_j\}_{j=1}^\infty$ are the orthonormal basis in Proposition 3.1. Let W_n be the space spanned by $\{w_1, w_2, \cdots, w_n\}$, and χ_n is the orthogonal projector in $\mathcal{L}^2$ into W_n. The approximating equation is as follows:

$$\begin{cases} \dfrac{d}{dt}u^\nu_n + \nu A u^\nu_n + \chi_n \mathbb{P}(u^\nu_n \cdot \nabla u^\nu_n) = 0, \\ \hfill u^\nu_n|_{t=0} = \chi_n a \end{cases} \tag{3.17}$$

Taking the L^2 inner product with $w_j, j = 1, 2, \ldots, n$, then the equation system (3.17) is equivalent with the following ordinary differential equation system:

$$\begin{cases} \dfrac{d}{dt}\alpha^\nu_{j,n}(t) + \nu\lambda_j \alpha^\nu_{j,n} + \displaystyle\sum_{k,\ell} b(\omega_k, \omega_\ell, \omega_j)\alpha^\nu_{k,n}\alpha^\nu_{\ell,n} = 0, \quad j = 1, 2, \ldots, n, \\ \hfill \alpha^\nu_{j,n}(0) = \left(a, \omega_j\right), \end{cases} \tag{3.18}$$

where $b(\omega_k, \omega_\ell, \omega_j) = \left(\omega_k \cdot \nabla\omega_\ell, \omega_j\right)$ satisfying $b(\omega_k, \omega_\ell, \omega_j) = -b(\omega_k, \omega_j, \omega_\ell)$. By the standard ordinary differential equation theory, there exists a solution to (3.18) local in time interval $[0, T_n)$. In order to show that T_n can be extended to T, we multiply with $\alpha^\nu_{j,n}(t)$ on both sides of (3.18) and take sum over $1 \leq j \leq n$. We have

$$\frac{1}{2}\frac{d}{dt}\left(\sum_{j=1}^n \alpha^\nu_{j,n}(t)^2\right) + \nu\lambda_j \sum_j \alpha^\nu_{j,n}(t)^2 = 0, \tag{3.19}$$

because

$$\sum_{j,k,\ell} b(\omega_k, \omega_\ell, \omega_j)\alpha^\nu_{k,n}\alpha^\nu_{\ell,n}\alpha^\nu_{j,n} = 0.$$

Moreover, from (3.19), we have

$$\left\|u_n^\nu(t,\cdot)\right\|_{L^2} = \sum_{j=1}^{n} \alpha_{j,n}^\nu(t)^2 \le \sum_{j=1}^{n} \alpha_{j,n}^\nu(0)^2 \le \|a\|_{L^2}^2, \quad \forall t > 0$$

Then, we have for every T_n; it can be extended to arbitrary large, so it can be extended to T. And we also obtain

$$u_n^\nu \text{ remains bounded in } L^\infty\left([0,T];L^2\right), \quad \forall n.$$

Moreover, we obtain a solution $u_n^\nu(t,x) = \sum_{j=1}^n \alpha_{j,n}^\nu(t) w_j(x)$ to (3.17), and we know that $u_n^\nu(t,x) \in \mathcal{G}^s_{r+1,\tau(t)}$ for $0 < t < T$ because it is only finite sum of w_j for fixed n. We then want to obtain the uniform Gevrey class norm bound for u_n^ν. To this end, we first apply $\Lambda^r e^{\tau\Lambda^{1/s}}$ on both sides of (3.17) and then take the L^2 inner product with $\Lambda^r e^{\tau\Lambda^{1/s}}$ to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|u_n^\nu(t,\cdot)\right\|^2_{\mathcal{G}^s_{r,\tau(t)}} + \nu\left\|u_n^\nu(t,\cdot)\right\|^2_{\mathcal{G}^s_{r+1,\tau}} \\
&\quad = \tau'(t)\left\|u_n^\nu(t,\cdot)\right\|^2_{\mathcal{G}^s_{r+\frac{1}{2s},\tau}} - \left(\Lambda^r e^{\tau\Lambda^{1/s}}\chi_n\mathbb{P}(u_n^\nu\cdot\nabla u_n^\nu), \Lambda^r e^{\tau(t)\Lambda^{1/s}} u_n^\nu\right).
\end{aligned}$$

We note that the operator χ_n and $\mathbb{P}$ commute with $\Lambda^r e^{\tau\Lambda^{1/s}}$, and they are symmetric, then

$$\left(\Lambda^r e^{\tau\Lambda^{1/s}}\chi_n\mathbb{P}(u_n^\nu\cdot\nabla u_n^\nu), \Lambda^r e^{\tau(t)\Lambda^{1/s}} u_n^\nu\right) = \left(\Lambda^r e^{\tau\Lambda^{1/s}}(u_n^\nu\cdot\nabla u_n^\nu), \Lambda^r e^{\tau(t)\Lambda^{1/s}} u_n^\nu\right).$$

With the arguments in Proposition 3.2, we have,

$$\|u_\nu^n(t,\cdot)\|_{\mathcal{G}^s_{r,\tau(t)}} \le G_T, \quad 0 < t \le T < T^*, \quad \forall n.$$

Thus,

$$u_n^\nu \text{ remains bounded in } L^\infty([0,T];\mathcal{G}^s_{r,\tau(\cdot)}). \tag{3.20}$$

In order to pass to the limit in the nonlinear term using a compactness theorem, we need to estimate on $\frac{du_n^\nu}{dt}$. From (3.17), we have

$$\frac{du_n^\nu}{dt} = -\nu A u_n^\nu - \chi_n\mathbb{P}(u_n^\nu\cdot\nabla u_n^\nu).$$

Then, we obtain

$$\begin{aligned}
\left\|\frac{du_n^\nu}{dt}\right\|_{L^2} &\le \left\|\chi_n\mathbb{P}(u_n^\nu\cdot\nabla u_n^\nu)\right\|_{L^2} + \left\|\nu A u_n^\nu\right\|_{L^2} \\
&\le \left\|u_n^\nu\cdot\nabla u_n^\nu\right\|_{L^2} + \nu_0\left\|u_n^\nu\right\|_{H^r} \\
&\le C\left\|u_n^\nu\right\|^2_{H^r} + \nu_0\left\|u_n^\nu\right\|_{H^r}.
\end{aligned}$$

We recall that

$$\left\|u_n^\nu(t,\cdot)\right\|_{H^r} \le C_T, \quad 0 < t \le T, \quad \forall n.$$

So, we obtain

$$\frac{d}{dt}u_m^\nu \text{ remains bounded in } L^\infty([0,T];L^2). \tag{3.21}$$

By (3.20) and (3.21), noting that $H^r(\mathbb{T}^3)$ is compactly embedded in $L^2(\mathbb{T}^3)$ from Rellich–Kondrachov Compactness Theorem in [34], a compactness theorem in [32, 33] indicates the existence of the limit $u^\nu \in L^\infty([0,T];\mathcal{G}^s_{r,\tau(\cdot)})$ of a subsequence of u_m^ν such that

$$\begin{cases} \frac{d}{dt}(u^\nu(t),v) - \nu(\Delta u^\nu(t),v) + (u^\nu(t)\cdot\nabla u^\nu(t),v) = 0, \;\; \forall v\in\mathcal{L}^2 \\ u^\nu(0) = a \end{cases} \tag{3.22}$$

For Euler equations, one can take very similar approach to obtain the existence of solution in Gevrey class space, and we omit the details here. Thus, we prove Proposition 3.1. □

It remains to show that u^ν is the solution of (1.1). In fact it can be obtained from (3.22) that

$$\mathbb{P}\left\{\frac{d}{dt}u^\nu(t) - \nu\Delta u^\nu(t) + u^\nu(t)\cdot\nabla u^\nu(t)\right\} = 0.$$

So, there exists a scalar function p^ν such that

$$\frac{d}{dt}u^\nu(t) - \nu\Delta u^\nu(t) + u^\nu(t)\cdot\nabla u^\nu(t) + \nabla p^\nu = 0,$$

where p^ν is unique up to a constant, and p^ν satisfies

$$-\Delta p^\nu = \nabla\cdot(u^\nu\cdot\nabla u^\nu), \tag{3.23}$$

with periodic boundary condition. For the regularity of the pressure $p^\nu(t,x)$, we have the following Proposition.

Proposition 3.3
Let p^ν satisfies (3.23) , then the following estimate holds,

$$\left\|p^\nu(t,\cdot)\right\|_{\mathcal{G}^s_{r+1,\tau}} \le CG_T^2, \quad 0<t\le T.$$

And for the pressure $p(t,x)$ in (1.2), we also have

$$\|p(t,\cdot)\|_{\mathcal{G}^s_{r+1,\tau}} \le CG_T^2, \quad 0<t\le T,$$

where C, T, G_T are defined in Proposition 3.2.

Proof
For us to study the pressure p^ν, the existence is obvious results from standard elliptic equation theory. We consider the regularity of $p^\nu(t,x)$. First, we apply the operator $\Lambda^r e^{\tau\Lambda^{1/s}}$ on both sides of (3.23) and then take L^2 inner product with $\Lambda^r e^{\tau\Lambda^{1/s}}p^\nu$ to obtain

$$-\left(\Delta p^\nu, \Lambda^{2r}e^{2\tau\Lambda^{1/s}}p^\nu\right) = \left(\nabla\cdot(u^\nu\cdot\nabla u^\nu), \Lambda^{2r}e^{2\tau\Lambda^{1/s}}p^\nu\right). \tag{3.24}$$

Here, if we can write $p^\nu(t,x) = \sum_{j\in\mathbb{Z}^3}\hat{p}_j^\nu e^{ij\cdot x}$, then the left side of (3.24) is

$$-\left(\Delta p^\nu, \Lambda^{2r}e^{2\tau\Lambda^{1/s}}p^\nu\right) = \sum_{j\in\mathbb{Z}^3}|j|^{2r+2}e^{2\tau|j|^{1/s}}\left|\hat{p}_j^\nu\right|^2 = \|p^\nu\|^2_{\mathcal{G}^s_{r+1,\tau}}.$$

The right hand side of (3.24) can be bounded by

$$\begin{aligned}
&\left|\left(\nabla\cdot(u^\nu\cdot\nabla u^\nu), \Lambda^{2r}e^{2\tau\Lambda^{1/s}}p^\nu\right)\right| \\
=&\left|(2\pi)^3\sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}|k|^{r-1}\left[\hat{u}_j^\nu\cdot(k-j)\right](k\cdot\hat{u}^\nu_{k-j})\hat{p}^\nu_{-k}\,|k|^{r+1}e^{2\tau|k|^{1/s}}\right| \\
=&\left|(2\pi)^3\sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}|k|^{r-1}\left[\hat{u}_j^\nu\cdot(k-j)\right](j\cdot\hat{u}^\nu_{k-j})\hat{p}^\nu_{-k}\,|k|^{r+1}e^{2\tau|k|^{1/s}}\right| \\
&\le C\left\|p^\nu\right\|_{\mathcal{G}^s_{r+1,\tau}} \\
&\times\left\{\sum_{k\in\mathbb{Z}^3}\left[\sum_{j\in\mathbb{Z}^3}(|j|^{r-1}+|k-j|^{r-1})\,|j|\,|k-j|\left|\hat{u}_j^\nu\right|e^{\tau|j|^{1/s}}e^{\tau|k|^{1/s}}\left|\hat{u}^\nu_{k-j}\right|\right]^2\right\}^{1/2} \\
&\le C\left\|u^\nu\right\|^2_{\mathcal{G}^s_{r,\tau}}\left\|p^\nu\right\|_{\mathcal{G}^s_{r+1,\tau}},
\end{aligned}$$

where C is a constant depending on r. Then from the aforementioned estimate, we obtain

$$\left\|p^\nu\right\|_{\mathcal{G}^s_{r+1,\tau}} \le C\left\|u^\nu\right\|^2_{\mathcal{G}^s_{r,\tau}}.$$

From (3.8), we obtain

$$\left\|p^\nu(t)\right\|_{\mathcal{G}^s_{r+1,\tau}} \le CA(t)^2 \le CG_T^2,\ t\in[0,T].$$

For the pressure $p(t,x)$ of Euler Eq. (1.2), one can first obtain the following elliptic equation,

$$-\Delta p = \nabla\cdot(u\cdot\nabla u).$$

Then using the same arguments as earlier, one can obtain the same results for p. □

4. Convergence of solutions in Gevrey space

In the previous section, we have proved the existence of solutions to the Navier–Stokes equation and Euler equation in Gevrey class space. In this section, we will show the vanishing viscosity limit of Navier–Stokes equation in Gevrey class space. Moreover, we can obtain the converging rate with respect to ν.

Theorem 4.1
Let u^ν, p^ν and u, p are the solutions we have obtained in the previous section, where

$$u^\nu, u \in L^\infty([0,T], \mathcal{G}^s_{r,\tau(\cdot)}), \quad p^\nu, p \in L^\infty([0,T], \mathcal{G}^s_{r+1,\tau(\cdot)}).$$

Then, the following estimates hold

$$\left\| u^\nu(t,\cdot) - u(t,\cdot) \right\|_{\mathcal{G}^s_{r-1,\tau(t)}} \le C\sqrt{\nu}, \quad \left\| p^\nu(t,\cdot) - p(t,\cdot) \right\|_{\mathcal{G}^s_{r,\tau(t)}} \le C\sqrt{\nu}, \tag{4.1}$$

for any $0 < t \le T$, where C is a constant depending on r, s, a, T.

Proof
Let us first consider the new equation for $w = (u^\nu - u)$ and $\tilde{p} = p^\nu - p$,

$$\begin{cases} \frac{\partial w}{\partial t} - \nu\Delta u^\nu + w\cdot\nabla u^\nu + u\cdot\nabla w + \nabla\tilde{p} &= 0, \\ \nabla\cdot w &= 0, \\ w|_{t=0} &= 0. \end{cases} \tag{4.2}$$

Then, we apply the operator $\Lambda^{r-1}e^{\tau\Lambda^{1/s}}$ on both sides of (4.2) and take the L^2 inner product with $\Lambda^{(r-1)}e^{\tau\Lambda^{1/s}}w$ on both sides to obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{\mathcal{G}^s_{r-1,\tau}} &= \nu\left(\Lambda^{(r-1)}e^{\tau\Lambda^{1/s}}\Delta u^\nu, \Lambda^{(r-1)}e^{\tau\Lambda^{1/s}}w\right) + \tau'\|w\|^2_{\mathcal{G}^s_{r-1+\frac{1}{2s},\tau}} \\ &\quad - \left(\Lambda^{(r-1)}e^{\tau\Lambda^{1/s}}(w\cdot\nabla u^\nu), \Lambda^{(r-1)}e^{\tau\Lambda^{1/s}}w\right) \\ &\quad - \left(\Lambda^{(r-1)}e^{\tau\Lambda^{1/s}}(u\cdot\nabla w), \Lambda^{(r-1)}e^{\tau\Lambda^{1/s}}w\right), \end{aligned} \tag{4.3}$$

where the term $\left(\Lambda^{(r-1)}e^{\tau\Lambda^{1/s}}\nabla\tilde{p},\, \Lambda^{(r-1)}e^{\tau\Lambda^{1/s}}w\right)$ disappeares because $w = u^\nu - u$ is divergence free. It remains to estimate the right hand side of (4.3); for convenience, we denote

$$\begin{aligned} \mathcal{I}_1 &= \nu\left(\Lambda^{(r-1)}e^{\tau\Lambda^{1/s}}\Delta u^\nu, \Lambda^{(r-1)}e^{\tau\Lambda^{1/s}}w\right), \\ \mathcal{I}_2 &= \left(\Lambda^{(r-1)}e^{\tau\Lambda^{1/s}}(w\cdot\nabla u^\nu), \Lambda^{(r-1)}e^{\tau\Lambda^{1/s}}w\right), \\ \mathcal{I}_3 &= \left(\Lambda^{(r-1)}e^{\tau\Lambda^{1/s}}(u\cdot\nabla w), \Lambda^{(r-1)}e^{\tau\Lambda^{1/s}}w\right). \end{aligned}$$

Using the discrete Hölder inequality, one can obtain

$$\begin{aligned} |\mathcal{I}_1| &= \nu\left|\left(\Lambda^{(r-1)}e^{\tau\Lambda^{1/s}}\Delta u^\nu, \Lambda^{(r-1)}e^{\tau\Lambda^{1/s}}w\right)\right| \\ &= \nu\left|-(2\pi)^3\sum_{k\in\mathbb{Z}^3}|k|^{2r}e^{2\tau|k|^{1/s}}(\hat{u}^\nu_k\cdot\hat{w}_{-k})\right| \\ &\le \nu(2\pi)^3\sum_{k\in\mathbb{Z}^3}|k|^{r+1}e^{\tau|k|^{1/s}}\left|\hat{u}^\nu_k\right||k|^{r-1}e^{\tau|k|^{1/s}}\left|\hat{w}_{-k}\right| \\ &\le C\nu\left\|u^\nu\right\|_{\mathcal{G}^s_{r+1,\tau}}\|w\|_{\mathcal{G}^s_{r-1,\tau}}. \end{aligned} \tag{4.4}$$

Then, we have

$$|\mathcal{I}_1| \le C\nu\left\|u^\nu\right\|_{\mathcal{G}^s_{r+1,\tau}}\|w\|_{\mathcal{G}^s_{r-1,\tau}}.$$

As for $\mathcal{I}_2$, we first write it into the sum of their Fourier coefficients,

$$\mathcal{I}_2 = i(2\pi)^3\sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}\left[\hat{w}_j\cdot(k-j)\right](\hat{u}^\nu_{k-j}\cdot\hat{w}_{-k})|k|^{2(r-1)}e^{2\tau|k|^{1/s}}.$$

Because $r > 9/2$, there exists a constant C such that

$$|k|^{r-1} \le C\left(|j|^{r-1} + |k-j|^{r-1}\right),$$

and $s \ge 1$ implies

$$e^{\tau|k|^{1/s}} \le e^{\tau|j|^{1/s}}e^{\tau|k-j|^{1/s}}.$$

Thus, $\mathcal{I}_2$ can be bounded by

$$|\mathcal{I}_2| \le C \sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}\Big[(|j|^{r-1}+|k-j|^{r-1})\left|\hat{w}_j\right||k-j|\left|\hat{u}^{\nu}_{k-j}\right| e^{\tau|j|^{1/s}}e^{\tau|k-j|^{1/s}} \times \left|\hat{w}_{-k}\right||k|^{r-1}e^{\tau|k|^{1/s}}\Big].$$

Then by discrete Hölder inequality and Minkowski inequality, we obtain

$$|\mathcal{I}_2| \le C\left\|u^{\nu}\right\|_{\mathcal{G}^s_{r,\tau}}\|w\|^2_{\mathcal{G}^s_{r-1,\tau}}. \tag{4.5}$$

As for $\mathcal{I}_3$, where

$$\mathcal{I}_3 = i(2\pi)^3\sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}[\hat{u}_j\cdot(k-j)](\hat{w}_{k-j}\cdot\hat{w}_{-k})\,|k|^{2(r-1)}\,e^{2\tau|k|^{1/s}}.$$

Here again, the cancelation property implies

$$\begin{aligned}0 &= \left(u\cdot\nabla\Lambda^{r-1}e^{\tau\Lambda^{1/s}}w,\Lambda^{r-1}e^{\tau\Lambda^{1/s}}w\right)\\ &= i(2\pi)^3\sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}[\hat{u}_j\cdot(k-j)](\hat{w}_{k-j}\cdot\hat{w}_{-k})\,|k-j|^{r-1}\,e^{\tau|k-j|^{1/s}}\,|k|^{r-1}\,e^{\tau|k|^{1/s}}.\end{aligned}$$

Then, we have

$$\begin{aligned}\mathcal{I}_3 &= i(2\pi)^3\sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}[\hat{u}_j\cdot(k-j)](\hat{w}_{k-j}\cdot\hat{w}_{-k})\,|k|^{2(r-1)}\,e^{2\tau|k|^{1/s}}\\ &\quad - i(2\pi)^3\sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}[\hat{u}_j\cdot(k-j)](\hat{w}_{k-j}\cdot\hat{w}_{-k})\,|k-j|^{r-1}\,e^{\tau|k-j|^{1/s}}\,|k|^{(r-1)}\,e^{\tau|k|^{1/s}}\\ &= \mathcal{R}_1+\mathcal{R}_2,\end{aligned}$$

where we denote

$$\mathcal{R}_1 = i(2\pi)^3\sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}\Big\{[\hat{u}_j\cdot(k-j)](\hat{w}_{k-j}\cdot\hat{w}_{-k})\left(|k|^{r-1}-|k-j|^{r-1}\right)e^{\tau|k|^{1/s}} \times|k|^{r-1}\,e^{\tau|k|^{1/s}}\Big\},$$

and

$$\mathcal{R}_2 = i(2\pi)^3\sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}\Big\{[\hat{u}_j\cdot(k-j)](\hat{w}_{k-j}\cdot\hat{w}_{-k})\,|k-j|^{r-1}\left(e^{\tau|k|^{1/s}}-e^{\tau|k-j|^{1/s}}\right) \times|k|^{r-1}\,e^{\tau|k|^{1/s}}\Big\}.$$

Here, we used a different strategy in the split of $|k|^{r-1}\,e^{\tau|k|^{1/s}}-|k-j|^{r-1}\,e^{\tau|k-j|^{1/s}}$ as in Lemma 2.3 to estimate $\mathcal{R}_1$ and $\mathcal{R}_2$. With use of the following mean value theorem, there exists a constant $\theta\in(0,1)$ such that

$$\begin{aligned}\left||k|^{r-1}-|k-j|^{r-1}\right| &= \left|(r-1)(|k|-|k-j|)\big[\theta\,|k|+(1-\theta)\,|k-j|\,\big]^{r-2}\right|\\ &\le C\,|j|\,(|k|^{r-2}+|k-j|^{r-2}).\end{aligned}$$

Then, by discrete Hölder inequality and Minkowski inequality, we have

$$\begin{aligned}|\mathcal{R}_1| &\le C\sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}\Big[\left|\hat{u}_j\right||k-j|\left|\hat{w}_{k-j}\right|\left|\hat{w}_{-k}\right||j|\,(|j|^{r-2}+|k-j|^{r-2})\\ &\qquad\times e^{\tau|j|^{1/s}}e^{\tau|k-j|^{1/s}}\,|k|^{r-1}\,e^{\tau|k|^{1/s}}\Big]\\ &\le C\,\|u\|_{\mathcal{G}^s_{r,\tau}}\|w\|^2_{\mathcal{G}^s_{r-1,\tau}}.\end{aligned}$$

As for $\mathcal{R}_2$, we use the inequality $\left|e^{\xi}-1\right|\le|\xi|\,e^{|\xi|}$ for $\forall\xi\in\mathbb{R}$ and Lemma 2.2,

$$|\mathcal{R}_2| \le C \sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}\Big[|\hat{u}_j|\,|\hat{w}_{k-j}|\,|\hat{w}_{-k}|\,|k-j|^r e^{\tau|k-j|^{1/s}} \left| e^{\tau(|k|^{1/s}-|k-j|^{1/s})}-1\right|$$
$$\times |k|^{r-1} e^{\tau|k|^{1/s}}\Big]$$
$$\le C\tau \sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}\Big[|\hat{u}_j|\,|\hat{w}_{k-j}|\,|\hat{w}_{-k}|\,|k-j|^{r-1} e^{\tau|k-j|^{1/s}} e^{\tau|j|^{1/s}} \frac{|j|\,|k-j|}{|k|^{1-\frac{1}{s}}+|k-j|^{1-\frac{1}{s}}}$$
$$\times |k|^{r-1} e^{\tau|k|^{1/s}}\Big].$$

For here, we use the inequality $|k-j| \le 2\,|k|\,|j|$ for $k, j \ne 0$, then we have

$$\frac{|k-j|}{|k|^{1-\frac{1}{s}}+|k-j|^{1-\frac{1}{s}}} \le |k-j|^{1/s}$$
$$\le C\,|k-j|^{\frac{1}{2s}}\,|k|^{\frac{1}{2s}}\,|j|^{\frac{1}{2s}},$$

where C is a constant depending on s. Thus, $\mathcal{R}_2$ can be bounded as follows,

$$|\mathcal{R}_2| \le C\tau \sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}\Big[|\hat{u}_j|\,|\hat{w}_{k-j}|\,|\hat{w}_{-k}|\,|k-j|^{r-1+\frac{1}{2s}} e^{\tau|k-j|^{1/s}} e^{\tau|j|^{1/s}}\,|j|^{1+\frac{1}{2s}}$$
$$\times |k|^{r-1+\frac{1}{2s}} e^{\tau|k|^{1/s}}\Big]$$
$$\le C\tau \sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}\Big[|\hat{u}_j|\,|\hat{w}_{k-j}|\,|\hat{w}_{-k}|\,|k-j|^{r-1+\frac{1}{2s}} e^{\tau|k-j|^{1/s}} (1+\tau\,|j|^{1/s} e^{|j|^{1/s}})$$
$$\times |j|^{1+\frac{1}{2s}}\,|k|^{r-1+\frac{1}{2s}} e^{\tau|k|^{1/s}}\Big]$$
$$\le C\tau\,\|u\|_{H^r}\,\|w\|^2_{\mathcal{G}^s_{r-1+\frac{1}{2s},\tau}} + C\tau^2\,\|u\|_{\mathcal{G}^s_{r,\tau}}\,\|w\|^2_{\mathcal{G}^s_{r-1+\frac{1}{2s},\tau}},$$

where we use the inequality $e^{\xi} \le 1+\xi e^{\xi}$ for $\forall \xi \in \mathbb{R}^+$ with respect to $e^{\tau|j|^{1/s}}$ and also the discrete Hölder inequality and Minkowski inequality in the earlier inequality. Then, we have

$$|\mathcal{I}_3| \le C\,\|u\|_{\mathcal{G}^s_{r,\tau}}\,\|w\|^2_{\mathcal{G}^s_{r-1,\tau}} + C\tau\,\|u\|_{H^r}\,\|w\|^2_{\mathcal{G}^s_{r-1+\frac{1}{2s},\tau}} + C\tau^2\,\|u\|_{\mathcal{G}^s_{r,\tau}}\,\|w\|^2_{\mathcal{G}^s_{r-1+\frac{1}{2s},\tau}}. \tag{4.6}$$

Substituting (4.4), (4.5) and (4.6) into (4.3), we obtain

$$\frac{1}{2}\frac{d}{dt}\|w\|^2_{\mathcal{G}^s_{r-1,\tau}} \le \nu\,\|u^\nu\|_{\mathcal{G}^s_{r+1,\tau}}\,\|w\|_{\mathcal{G}^s_{r-1,\tau}} + C\,\|u\|_{\mathcal{G}^s_{r,\tau}}\,\|w\|^2_{\mathcal{G}^s_{r-1,\tau}}$$
$$+\left(\tau' + C\tau\,\|u\|_{H^r} + C\tau^2\,\|u\|_{\mathcal{G}^s_{r,\tau}}\right)\|w\|^2_{\mathcal{G}^s_{r-1+\frac{1}{2s},\tau}}. \tag{4.7}$$

By the choice of τ in (3.13), and noting that (3.8),(3.9),(3.10),(3.11) also hold for Euler equation (1.2). Then choosing the appropriate constant C, one has $\tau' + C\tau\,\|u\|_{H^r} + C\tau^2\,\|u\|_{H^r} + C\tau^2\,\|u\|_{\mathcal{G}^s_{r,\tau}} \le 0$; then we can obtain from (4.7) and (3.12),

$$\frac{d}{dt}\|w(t,\cdot)\|_{\mathcal{G}^s_{r-1,\tau(t)}} \le \nu\,\|u^\nu(t,\cdot)\|_{\mathcal{G}^s_{r+1,\tau(t)}} + CG_T\,\|w(t,\cdot)\|_{\mathcal{G}^s_{r-1,\tau(t)}}, \quad 0 < t \le T. \tag{4.8}$$

Because $w(0) = 0$ and Grownwall's inequality, (4.8) implies

$$\|w(t,\cdot)\|_{\mathcal{G}^s_{r-1,\tau(t)}} \le e^{CG_T t}\int_0^t \nu\,\|u^\nu(s,\cdot)\|_{\mathcal{G}^s_{r+1,\tau(s)}}\,ds, \quad 0 < t \le T.$$

Recalling from (3.16) we have for $0 < t \le T$,

$$\int_0^t \nu\,\|u^\nu(s,\cdot)\|^2_{\mathcal{G}^s_{r+1,\tau(s)}}\,ds \le M_T, \;\; t \in (0, T].$$

With Hölder inequality, we have

$$\int_0^t \nu\,\|u^\nu(s,\cdot)\|_{\mathcal{G}^s_{r+1,\tau(s)}}\,ds \le \int_0^t \sqrt{\nu}\sqrt{\nu}\,\|u^\nu(s,\cdot)\|_{\mathcal{G}^s_{r+1,\tau(s)}}\,ds$$
$$\le \sqrt{\nu}\,t^{1/2} M_T^{1/2}.$$

Then, we have

$$\|w(t,\cdot)\|_{\mathcal{G}^s_{r-1,\tau(t)}} \le C\sqrt{\nu}M_T^{1/2}t^{1/2}e^{CG_Tt},\ \ 0<t\le T. \tag{4.9}$$

This proves the first estimate of (4.1) by arranging the constant. Then, we want to estimate $p^\nu(t)-p(t)$ in the norm of $G^s_{r,\tau}$. To do so, we first take the divergence of both sides of (4.2) to obtain the following elliptic equation:

$$-\Delta\tilde{p} = \nabla\cdot(w\cdot\nabla u^\nu) + \nabla\cdot(u\cdot\nabla w). \tag{4.10}$$

Then, we first apply the operator $\Lambda^{r-1}e^{\tau\Lambda^{1/s}}$ on both sides of (4.10) and then take the L^2 inner product with $\Lambda^{(r-1)}e^{\tau\Lambda^{1/s}}\tilde{p}$ on both sides to obtain

$$\begin{aligned}(2\pi)^3\|\tilde{p}\|^2_{\mathcal{G}^s_{r,\tau}} &= \left(\Lambda^{r-1}e^{\tau\Lambda^{1/s}}\nabla\cdot(w\cdot\nabla u_\nu),\Lambda^{(r-1)}e^{\tau\Lambda^{1/s}}\tilde{p}\right)\\ &\quad+\left(\Lambda^{r-1}e^{\tau\Lambda^{1/s}}\nabla\cdot(u\cdot\nabla w),\Lambda^{(r-1)}e^{\tau\Lambda^{1/s}}\tilde{p}\right)\\ &= i^2(2\pi)^3\sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}[\hat{w}_j\cdot(k-j)](k\cdot\hat{u}^\nu_{k-j})\,|k|^{2(r-1)}\,e^{2\tau|k|^{1/s}}\hat{\tilde{p}}_{-k}\\ &\quad+i^2(2\pi)^3\sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}[\hat{u}_j\cdot(k-j)](k\cdot\hat{w}_{k-j})\,|k|^{2(r-1)}\,e^{2\tau|k|^{1/s}}\hat{\tilde{p}}_{-k}\\ &= \mathcal{P}_1+\mathcal{P}_2,\end{aligned} \tag{4.11}$$

where we denote

$$\begin{aligned}\mathcal{P}_1 &= i^2(2\pi)^3\sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}[\hat{w}_j\cdot(k-j)](k\cdot\hat{u}^\nu_{k-j})\,|k|^{2(r-1)}\,e^{2\tau|k|^{1/s}}\hat{\tilde{p}}_{-k},\\ \mathcal{P}_2 &= i^2(2\pi)^3\sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}[\hat{u}_j\cdot(k-j)](k\cdot\hat{w}_{k-j})\,|k|^{2(r-1)}\,e^{2\tau|k|^{1/s}}\hat{\tilde{p}}_{-k}.\end{aligned}$$

It remains to estimate $\mathcal{P}_1$ and $\mathcal{P}_2$ in (4.11). For simplicity, we only compute $\mathcal{P}_1$; because $\mathcal{P}_2$ can be estimated in the same way.

$$\begin{aligned}|\mathcal{P}_1| &= \left|(2\pi)^3\sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}[\hat{w}_j\cdot(k-j)](j\cdot\hat{u}^\nu_{k-j})\,|k|^{2(r-1)}\,e^{2\tau|k|^{1/s}}\hat{\tilde{p}}_{-k}\right|\\ &\le (2\pi)^3\sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}\Big[C(|j|^{r-2}+|k-j|^{r-2})e^{\tau|j|^{1/s}}e^{\tau|k-j|^{1/s}}\,|k-j|\\ &\quad\times|j|\left|\hat{w}_j\right|\left|\hat{u}^\nu_{k-j}\right||k|^r\,e^{\tau|k|^{1/s}}\left|\hat{\tilde{p}}_{-k}\right|\Big]\\ &\le C\|w\|_{G^s_{r-1,\tau}}\left\|u^\nu\right\|_{G^s_{r,\tau}}\|\tilde{p}\|_{G^s_{r,\tau}},\end{aligned} \tag{4.12}$$

where we use the fact $k\cdot\hat{u}^\nu_{k-j} = j\cdot\hat{u}^\nu_{k-j}$ from the divergence free condition. And, similarly, we can obtain

$$\begin{aligned}|\mathcal{P}_2| &\le (2\pi)^3\sum_{k\in\mathbb{Z}^3}\sum_{j\in\mathbb{Z}^3}\left|[\hat{u}_j\cdot(k-j)](j\cdot\hat{w}_{k-j})\,|k|^{2(r-1)}\,e^{2\tau|k|^{1/s}}\hat{\tilde{p}}_{-k}\right|\\ &\le C\|w\|_{\mathcal{G}^s_{r-1,\tau}}\|u\|_{\mathcal{G}^s_{r,\tau}}\|\tilde{p}\|_{\mathcal{G}^s_{r,\tau}}.\end{aligned} \tag{4.13}$$

Substituting (4.12) and (4.13) into (4.11), we obtain

$$\|\tilde{p}(t,\cdot)\|_{\mathcal{G}^s_{r,\tau(t)}} \le CG_T\|w(t,\cdot)\|_{\mathcal{G}^s_{r-1,\tau(t)}},\quad 0<t\le T. \tag{4.14}$$

Then by (4.9) and (4.14), we have

$$\|\tilde{p}(t,\cdot)\|_{\mathcal{G}^s_{r,\tau(t)}} \le C\sqrt{\nu}M_T^{1/2}t^{1/2}G_Te^{CG_Tt},\quad 0\le t\le T.$$

This proves (4.1) by arranging the constants. Thus, we have proven Theorem 1.1. □

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