SIAM J. MATH. ANAL.
Vol. 48, No. 3, pp. 1672–1726

GEVREY CLASS SMOOTHING EFFECT FOR THE PRANDTL EQUATION*

WEI-XI LI†, DI WU‡, AND CHAO-JIANG XU§

Abstract. It is well known that the Prandtl boundary layer equation is unstable for general initial data, and is well-posed in Sobolev space for monotonic initial data. Recently, under the Oleinik's monotonicity assumption for the initial datum, R. Alexandre, Y. Wang, C.-J. Xu, and T. Yang [*J. Amer. Math. Soc.*, 28 (2015) pp. 745–784] recovered the local well-posedness of the Cauchy problem in Sobolev space by virtue of an energy method (see also N. Masmoudi and T. K. Wong [*Comm. Pure Appl. Math.,* 68 (2015), pp. 1683–1741.]). In this work, we study the Gevrey smoothing effects of the local solution obtained in R. Alexandre, Y. Wang, C.-J. Xu, and T. Yang [*J. Amer. Math. Soc.*, 28 (2015) pp. 745–784]. We prove that the Sobolev's class solution belongs to some Gevrey class with respect to tangential variables at any positive time.

Key words. Prandtl's equation, Gevrey class, subelliptic estimate, monotonicity condition

AMS subject classifications. 35M13, 35Q35, 76D10, 76D03, 76N20

DOI. 10.1137/15M1020368

1. Introduction. In this work, we study the regularity of solutions to the Prandtl equation which is the foundation of the boundary layer theory introduced by Prandtl in 1904 [24]. The inviscid limit of an incompressible viscous flow with the nonslip boundary condition is still a challenging problem of mathematical analysis due to the appearance of a boundary layer, where the tangential velocity adjusts rapidly from nonzero at the area far away from the boundary to zero on the boundary. The Prandtl equation describes the behavior of the flow near the boundary as the viscosity is very small, and it reads

$$\begin{cases} u_t + uu_x + vu_y + p_x = u_{yy}, \quad t>0, \quad x\in\mathbb{R}, \quad y>0, \\ u_x + v_y = 0, \\ u|_{y=0} = v|_{y=0} = 0, \ \lim\limits_{y\to+\infty} u = U(t,x), \\ u|_{t=0} = u_0(x,y)\,, \end{cases}$$

where $u(t,x,y)$ and $v(t,x,y)$ represent the tangential and normal velocities of the boundary layer, with y being the scaled normal variable to the boundary, while $U(t,x)$ and $p(t,x)$ are the values on the boundary of the tangential velocity and pressure of the outflow satisfying the Bernoulli law

$$\partial_t U + U\partial_x U + \partial_x q = 0.$$

*Received by the editors May 7, 2015; accepted for publication (in revised form) March 21, 2016; published electronically May 5, 2016.
http://www.siam.org/journals/sima/48-3/M102036.html

†School of Mathematics and Statistics, and Computational Science Hubei Key Laboratory, Wuhan University, 430072 Wuhan, People's Republic of China (wei-xi.li@whu.edu.cn). The research of this author was supported by NSF of China (11422106).

‡School of Mathematics and Statistics, Wuhan University, 430072 Wuhan, People's Republic of China (wudi2530@whu.edu.cn).

§School of Mathematics and Statistics, Wuhan University, 430072 Wuhan, People's Republic China, and CNRS UMR 6085, Laboratoire de Mathématiques, Université de Rouen, 76801 Saint-Etienne du Rouvray, France (Chao-Jiang.Xu@univ-rouen.fr). The research of this author was partially supported by "the Fundamental Research Funds for the Central Universities" and the NSF of China (11171261).

Because of the degeneracy in the tangential variable, the well-posedness theories and the justification of the Prandtl's boundary layer theory remain as the challenging problems in the mathematical theory of fluid mechanics. Up to now, there are only a few rigorous mathematical results (see [4, 13, 14, 15, 22] and references therein). Under a monotonic assumption on the tangential velocity of the outflow, Oleinik was the first to obtain the local existence of classical solutions for the initial boundary value problems, and this result together with some of her works with collaborators were well presented in the monograph [23]. In addition to Oleinik's monotonicity assumption on the velocity field, by imposing a so-called favorable condition on the pressure, Xin and Zhang [26] obtained the existence of global weak solutions to the Prandtl equation. All these well-posedness results were based on the Crocco transformation to overcome the main difficulty caused by degeneracy and mixed type of the equation. Just recently the well-posedness in the Sobolev space was explored by virtue of an energy method instead of the Crocco transformation; see Alexandre et al. [1] and Masmoudi and Wong [21]. There are very few works concerned with the Prandtl equation without the monotonicity assumption; we refer to [2, 3, 20, 9, 25, 30] for the works in the analytic frame, and [12, 17] for the recent works concerned with the existence in the Gevrey class. Recall the Gevrey class, denoted by $G^s, s \geq 1$, is an intermediate space between analytic functions and C^∞ space. For a given domain Ω, the (global) Gevrey space $G^s(\Omega)$ consists of such functions that $f \in C^\infty(\Omega)$ and that

$$\left\|\partial^\alpha f\right\|_{L^2(\Omega)} \leq L^{|\alpha|+1}(\alpha!)^s$$

for some constant L independent of α. The significant difference between Gevrey ($s > 1$) and analytic ($s = 1$) classes is that there exist nontrivial Gevrey functions admitting compact support.

We mention that due to the degeneracy in x, it is natural to expect Gevrey regularity rather than analyticity for a subelliptic equation. We refer to [5, 6, 7, 8] for the link between subellipticity and Gevrey reguality. In this paper we first study the intrinsic subelliptic structure due to the monotonicity condition, and then deduce, based on the subelliptic estimate, the Gevrey smoothing effect; that is, given a monotonic initial data belonging to some Sobolev space, the solution will lie in some Gevrey class at positive time, just like the heat equation. It is different from the Gevrey propagation property obtained in the aforementioned works, where the initial data are supposed to be of some Gevrey class, for instance $G^{7/4}$ in [12], and the well-posedness is obtained in the same Gevrey space.

Now we state our main result. Without loss of generality, we only consider here the case of a uniform outflow $U = 1$, and the conclusion will still hold for Gevrey class outflow U. We mention that the Gevrey regularity for outflow U is well-developed (see [18] for instance). For the uniform outflow, we get the constant pressure p due to the Bernoulli law. Then the Prandtl equation can be rewritten as

$$(1.1) \qquad \begin{cases} u_t + uu_x + vu_y - u_{yy} = 0, \quad (t,x,y) \in]0,T[\times \mathbb{R}^2_+, \\ u_x + v_y = 0, \\ u|_{y=0} = v|_{y=0} = 0, \quad \lim\limits_{y\to+\infty} u = 1, \\ u|_{t=0} = u_0(x,y)\,. \end{cases}$$

The main result concerned with the Gevrey class regularity can be stated as follows.

THEOREM 1.1. *Let $u(t,x,y)$ be a classical local in time solution to Prandtl equation* (1.1) *on* $[0,T]$ *with the properties subsequently listed below:*

(i) *There exist two constants* $C_* > 1, \sigma > 1/2$ *such that for any* $(t,x,y) \in [0,T]\times\mathbb{R}^2_+$,

$$\begin{aligned} C_*^{-1}\langle y\rangle^{-\sigma} \le \partial_y u(t,x,y) \le C_*\langle y\rangle^{-\sigma}, \\ \left|\partial_y^2 u(t,x,y)\right| + \left|\partial_y^3 u(t,x,y)\right| \le C_*\langle y\rangle^{-\sigma-1}, \end{aligned} \tag{1.2}$$

where $\langle y\rangle = (1+|y|^2)^{1/2}$.

(ii) *There exists* $c > 0, C_0 > 0$ *and integer* $N_0 \ge 7$ *such that*

$$\|e^{2cy}\partial_x u\|_{L^\infty\left([0,T];\ H^{N_0}(\mathbb{R}^2_+)\right)} + \|e^{2cy}\partial_x\partial_y u\|_{L^2\left([0,T];\ H^{N_0}(\mathbb{R}^2_+)\right)} \le C_0. \tag{1.3}$$

Then for any $0 < T_1 < T$, *there exists a constant* L, *such that for any* $0 < t \le T_1$,

$$\forall\ m > 1+N_0, \quad \left\|e^{\tilde c y}\partial_x^m u(t)\right\|_{L^2(\mathbb{R}^2_+)} \le t^{-3(m-N_0-1)}\, L^m\, (m!)^{3(1+\sigma)}, \tag{1.4}$$

where $0 < \tilde c < c$. *The constant* L *depends only on* $C_0, T_1, C_*, c, \tilde c$, *and* σ. *Therefore, the solution* u *belongs to the Gevrey class of index* $3(1+\sigma)$ *with respect to* $x \in \mathbb{R}$ *for any* $0 < t \le T_1$.

Remark 1.2.

(1) The solution described in the above theorem exists; for instance, suppose that the initial data u_0 can be written as

$$u_0(x,y) = u_0^s(y) + \tilde u_0(x,y),$$

where u_0^s is a function of y but independent of x such that $C^{-1}\langle y\rangle^{-\sigma} \le \partial_y u_0^s(y) \le C\langle y\rangle^{-\sigma}$ for some constant $C \ge 1$, and $\tilde u_0$ is a small perturbation such that its weighted Sobolev norm $\left\|e^{2cy}\tilde u_0\right\|_{H^{2N_0+7}(\mathbb{R}^2_+)}$ is suitably small. Then using the arguments in [1], we can obtain the desired solution with the properties listed in Theorem 1.1 fulfilled. Precisely, the solution $u(t,x,y)$ is a perturbation of a shear flow $u^s(t,y)$ such that property (i) in the above theorem holds for u, and moreover $e^{2cy}(u-u^s) \in L^\infty\left([0,T];\ H^{N_0+1}(\mathbb{R}^2_+)\right)$. Moreover following the analysis in [21] with some modifications, we can also obtain more general solutions with exponential decay rather than perturbative solutions around monotonic shear flows.

(2) The well-posedness problem of Prandtl's equation depends crucially on the choice of the underlying function spaces, especially on the regularity in the tangential variable x. If the initial datum is analytic in x, then the local in time solution exists(c.f. [20, 25, 30]), but the Cauchy problem is ill-posed in Sobolev space for linear and nonlinear Prandtl equations (cf. [10, 11]). Indeed, the main mathematical difficulty is the lack of control the x derivatives. For example, v in (1.1) could be written as $-\int_0^y u_x(y')dy'$ by the divergence-free condition, and here we lose one derivatives in x-regularity. The degeneracy can't be balanced directly by any horizontal diffusion term, so that the standard energy estimates do not apply to establish the existence of local solution. But the results in our main Theorem 1.1 shows that *the loss of derivative in the tangential variable* x *can be partially compensated for via the monotonicity condition.*

(3) Under the hypothesis (1.2), (1.1) is a nonlinear hypoelliptical equation of Hörmander type with a gain of regularity of order $\frac{1}{3}$ in the x variable (see Proposition 2.4), so that any C^2 solution is locally C^∞ (see [27, 28, 29]); for the corresponding linear operator, [8] obtained the regularity in the local

Gevrey space G^3. However, in this paper we study (1.1) as a boundary layer equation, so that the local property of solution is not of interest to the physics application, and our goal is then to study the global estimates in Gevrey class. In view of (1.2) we see u_y decays polynomially at infinity, so we only have a weighted subelliptic estimate (see Proposition 2.4). This explains why the Gevrey index, which is $3(1+\sigma)$, depends also on the decay index σ in (1.2).

(4) Finally, the estimate (1.4) gives an explicit Gevrey norm of solutions for the Cauchy problem with respect to $t>0$ when the initial datum is only in some finite order Sobolev space. Since the Gevrey class is an intermediate space between analytic space and Sobolev space, the qualitative study of solutions in the Gevery class can help us to understand the Prandtl boundary layer theory which has been justified in the analytic frame.

The approach. We end our introduction by explaining the main idea used in the proof. It's well known that the main difficulty for the Prandtl equation is the degeneracy in the x variable, due to the presence of v:

$$v=-\int_0^y (\partial_x u)\,d\tilde{y}.$$

To overcome the degeneracy, we use the cancellation idea, introduced by Masmoudi and Wong [21], to perform the estimates on the new function and, moreover, on the original velocity function u. Precisely, observe

$$u_t+uu_x+vu_y-u_{yy}=0,$$

and, with $\omega=\partial_y u$,

$$\omega_t+u\omega_x+v\omega_y-\omega_{yy}=0.$$

In order to eliminate the v term on the left sides of the above two equations, we use the monotonicity condition $\partial_y u=\omega>0$ and thus multiply the second equation by $-\frac{\partial_y\omega}{\omega}$, and then add the resulting equation to the first one; this gives, denoting $f=\omega-\frac{\partial_y\omega}{\omega}u$,

$$f_t+u\partial_x f-\partial_{yy}f=\text{terms of lower order}.$$

Our main observation for the new equation is the intrinsic subelliptic structure due to the monotonicity condition. Indeed, denoting $X_0=\partial_t+u\partial_x$ and $X_1=\partial_y$, we can rewrite the above equation as Hörmander type:

$$\Big(X_0+X_1^*X_1\Big)f=\text{terms of lower order},$$

and, moreover, direct computation shows

$$\big[X_1,\ X_0\big]=(\partial_y u)\,\partial_x. \tag{1.5}$$

Thus Hörmander's bracket condition will be fulfilled, provided by $\partial_y u>0$, and consequently the following subelliptic estimate holds:

$$\forall\ w\in C_0^\infty(K),\quad \big\|\Lambda^{2/3}w\big\|_{L^2}\lesssim\big\|\Big(X_0+X_1^*X_1\Big)w\big\|_{L^2}+\big\|w\big\|_{L^2}$$

with K a compact subset of $\mathbb{R}^3_{t,x,y}$ and $\Lambda^d = \Lambda^d_x$ is the Fourier multiplier of symbol $\langle\xi\rangle^d$ with respect to $x \in \mathbb{R}$. We refer to [16] for details on the general subelliptic operator. We remark that the above subelliptic estimate is local, and as the far as the Prandtl equation is concerned, the situation is more complicated: on one side only the global estimate is interesting, that is, we have to consider $y \geq 0$ rather than a compacted subset of $\mathbb{R}_+$, on the other there are boundary and initial problems. When y varies in the half-line $y \geq 0$ the Hörmander's bracket condition (1.5) is no longer true, since $\partial_y u \to 0$ as $y \to +\infty$. To overcome this difficulty we perform, following the arguments used in the classical (local) subelliptic estimate with some modification, a weighted subelliptic estimate of the following form: for any $w \in L^2\left([0,T], H^2(\mathbb{R}^2_+)\right)$,

$$\left\| |\partial_y u|^{1/2} \Lambda^{1/3} w \right\|_{L^2} \lesssim \left\| \Big(X_0 + X_1^* X_1 \Big) w \right\|_{L^2} + \|w\|_{L^2} \\ + \text{terms from boundary conditions},$$

which indicates the loss-gain phenomenon; that is, in order to gain $\Lambda^{1/3}$ regularity we have to loss $|\partial_y u|^{1/2}$ weight. Similarly as far as the higher derivatives $\partial_x^m u$ are concerned, we can execute an equation for

$$f_m = \partial_x^m \omega - \frac{\partial_y \omega}{\omega} \partial_x^m u = \omega \partial_y \left(\frac{\partial_x^m u}{\omega} \right), \quad m \geq 1,$$

to cancel the bad term involving $\partial_x^m v$, and, moreover, the above weighted sublliptic estimate still holds for this equation. Moreover, by the Hardy inequality, in order to obtain the control of $\partial_x^m \omega$ and $\partial_x^m u$, it is sufficient to perform estimates on f_m (see section 4 for details).

Our choice of the weight function W_m^ℓ (see (2.2) below) is motivated by the loss-gain estimate. Recall

$$W_m^\ell = e^{2cy} \left(1 + \frac{2cy}{(3m+\ell)\sigma} \right)^{-\frac{(3m+\ell)\sigma}{2}} (1+cy)^{-1} \Lambda^{\frac{\ell}{3}}, \quad 0 \leq \ell \leq 3, \ \ m \in \mathbb{N}, \ \ y > 0,$$

where the essential part is the factor

$$\left(1 + \frac{2cy}{(3m+\ell)\sigma} \right)^{-\frac{(3m+\ell)\sigma}{2}} \Lambda^{\frac{\ell}{3}}.$$

Thus as ℓ is increased by one, we gain $\Lambda^{\frac{1}{3}}$ regularity and meanwhile loss the weight $\langle y \rangle^{-\frac{\sigma}{2}} \sim |\partial_y u|^{\frac{1}{2}}$. Moreover,

$$\left(1 + \frac{2cy}{(3m+\ell)\sigma} \right)^{-\frac{(3m+\ell)\sigma}{2}}$$

is bounded from below by e^{-cy} and goes to 0 as $y \to +\infty$, so we add the factor e^{2cy} in the expression of W_m^ℓ to guarantee the strictly positive lower bound. Another factor $(1+cy)^{-1}$ is introduced for the purpose that

$$\partial_y \left(e^{2cy} \left(1 + \frac{2cy}{(3m+\ell)\sigma} \right)^{-\frac{(3m+\ell)\sigma}{2}} (1+cy)^{-1} \right) \Bigg|_{y=0} = 0.$$

Observe that the Prandtl equation is an initial boundary problem, and we will study the smoothing effect. Thus it is natural to introduce a cutoff function in time:

$$\phi_m^\ell = \phi^{3(m-(N_0+1))+\ell} = (t(T-t))^{3(m-(N_0+1))+\ell}, \quad m \geq N_0+1,\ 0 \leq \ell \leq 3,$$

which ensures that $\phi_m^\ell f_m$ vanishes at the endpoints.

Now we perform the equation for $G_m^\ell = \phi_m^\ell W_m^\ell f_m$:

$$\begin{cases} \left(\partial_t + u\partial_x + v\partial_y - \partial_y^2\right) G_m^\ell = (\partial_t \phi_m^\ell) W_m^\ell f_m + \cdots, \\ \partial_y G_m^\ell\big|_{y=0} = 0, \\ G_m^\ell\big|_{t=0} = 0, \end{cases}$$

and have the energy estimate

$$\|G_m^\ell\|_{L^\infty\left([0,T];\ L^2(\mathbb{R}_+^2)\right)} + \left\|\partial_y G_m^\ell\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} \lesssim m^{1/2}\left\|\phi^{-1/2} G_m^\ell\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} + \cdots.$$

We have to control the first term on the right-hand side, which arises from the commutator between ∂_t and the cutoff function ϕ_m^ℓ and is crucial to study the smoothing effect. Here we will make use of the weighted subelliptic estimate (see section 3) to treat this term. More details can be found in section 2.

The paper is organized as follows. In section 2 we prove Theorem 1.1, and state some preliminary lemmas used in the proof. The other sections are occupied by the proof of the preliminaries lemmas. Precisely, we prove in section 3 a subelliptic estimate for the linearized Prandtl operator. Sections 4 and 5 are devoted to presenting a crucial estimate for an auxilliary function and nonlinear terms. The last section is an appendix, where the equation fulfilled by the auxilliary function is deduced.

2. Proof for the main theorem. We will prove in this section the Gevery estimate (1.4) by induction on m. As in [21], we consider the following auxilliary function

$$f_m = \partial_x^m \omega - \frac{\partial_y \omega}{\omega}\partial_x^m u = \omega \partial_y \left(\frac{\partial_x^m u}{\omega}\right), \quad m \geq 1, \tag{2.1}$$

where $\omega = \partial_y u > 0$ and u is a solution of (1.1) which satisfies the hypotheis (1.2). We also introduce the following inductive weight,

$$W_m^\ell = e^{2cy}\left(1 + \frac{2cy}{(3m+\ell)\sigma}\right)^{-\frac{(3m+\ell)\sigma}{2}} (1+cy)^{-1}\Lambda^{\frac{\ell}{3}}, \tag{2.2}$$
$$0 \leq \ell \leq 3,\ \ m \in \mathbb{N},\ y > 0,$$

where $\Lambda^d = \Lambda_x^d$ is the Fourier multiplier of symbol $\langle\xi\rangle^d$ with respect to $x \in \mathbb{R}$. Note

$$W_m^0 \geq e^{cy}(1+cy)^{-1} \geq c_0 e^{\tilde{c}y} \tag{2.3}$$

for $0 < \tilde{c} < c$.

Since

$$\left|\frac{\partial_y \omega}{\omega}\right| \leq C_*^2 \langle y\rangle^{-1},$$

we have that , if u is smooth,

$$\|W_m^0 f_m\|_{L^2(\mathbb{R}_+^2)} \leq \|W_m^0 \partial_x^m \omega\|_{L^2(\mathbb{R}_+^2)} + C_*^2 \|W_m^0 \langle y\rangle^{-1} \partial_x^m u\|_{L^2(\mathbb{R}_+^2)}.$$

On the other hand, we have the following Poincaré-type inequality.

Lemma 2.1. *There exist $C_1, \tilde{C}_1 > 0$ independent of $m \geq 1, 0 \leq \ell \leq 3$, such that*

$$\big\| \langle y \rangle^{-1} W_m^\ell \partial_x^m u \big\|_{L^2(\mathbb{R}_+^2)} + \big\| \langle y \rangle^{-1} W_m^\ell \partial_x^m \omega \big\|_{L^2(\mathbb{R}_+^2)} \leq C_1 \big\| W_m^\ell f_m \big\|_{L^2(\mathbb{R}_+^2)}. \tag{2.4}$$

As a result,

$$\big\| \Lambda^{-1} W_m^0 f_{m+1} \big\|_{L^2(\mathbb{R}_+^2)} \leq \tilde{C}_1 \big\| W_m^0 f_m \big\|_{L^2(\mathbb{R}_+^2)} \tag{2.5}$$

and

$$\big\| \Lambda^{-1} \partial_y W_m^0 f_{m+1} \big\|_{L^2(\mathbb{R}_+^2)} \leq \tilde{C}_1 \left(\big\| \partial_y W_m^0 f_m \big\|_{L^2(\mathbb{R}_+^2)} + \big\| W_m^0 f_m \big\|_{L^2(\mathbb{R}_+^2)} \right).$$

We will prove the above lemma in section 4 as Lemma 4.2.

Since the initial datum of (1.1) is only in the Sobolev space H^{N_0+1}, we have to introduce the following cutoff function, with respect to $0 \leq t \leq T \leq 1$, to study the Gevrey smoothing effect by using the hyopellipticity

$$\phi_m^\ell = \phi^{3(m-(N_0+1))+\ell} = (t(T-t))^{3(m-(N_0+1))+\ell}, \quad m \geq N_0 + 1,\ 0 \leq \ell \leq 3. \tag{2.6}$$

We will prove by induction an energy estimate for the function $\phi_m^0 W_m^0 f_m$. For this purpose we need the following lemma concerned with the link between $\phi_{m+1}^0 W_{m+1}^0 f_{m+1}$ and $\phi_m^3 W_m^3 f_m$, whose proof is postponed to section 4 as Lemmas 4.3 and 4.4.

Lemma 2.2. *There exists a constant C_2, depending only on the numbers σ, c, and the constant C_* in Theorem 1.1, in particular, independent of m, such that for any $m \geq N_0 + 1$,*

$$\begin{aligned} &\big\| \phi_{m+1}^0 W_{m+1}^0 f_{m+1} \big\|_{L^\infty([0,T];\ L^2(\mathbb{R}_+^2))} + \sum_{j=1}^2 \big\| \partial_y^j \Lambda^{-\frac{2(j-1)}{3}} \phi_{m+1}^0 W_{m+1}^0 f_{m+1} \big\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ &\quad \leq\ C_2 \big\| \phi_m^3 W_m^3 f_m \big\|_{L^\infty([0,T];\ L^2(\mathbb{R}_+^2))} + C_2 \sum_{j=1}^2 \big\| \partial_y^j \Lambda^{-\frac{2(j-1)}{3}} \phi_m^3 W_m^3 f_m \big\|_{L^2([0,T]\times\mathbb{R}_+^2)}, \end{aligned}$$

$$\begin{aligned} &\big\| \partial_y^3 \Lambda^{-1} \phi_{m+1}^0 W_{m+1}^0 f_{m+1} \big\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ &\quad \leq\ C_2 \big\| \phi_m^3 W_m^3 f_m \big\|_{L^\infty([0,T];\ L^2(\mathbb{R}_+^2))} + C_2 \sum_{j=1}^2 \big\| \partial_y^j \Lambda^{-\frac{2(j-1)}{3}} \phi_m^3 W_m^3 f_m \big\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ &\quad\quad + C_2 \big\| \partial_y^3 \Lambda^{-1} \phi_m^3 W_m^3 f_m \big\|_{L^2([0,T]\times\mathbb{R}_+^2)}, \end{aligned}$$

and

$$\begin{aligned} \big\| \langle y \rangle^{-\sigma/2} \partial_y \Lambda^{1/3} \Lambda_\delta^{-2} W_m^{\ell-1} f_m \big\|_{L^2(\mathbb{R}_+^2)} \leq C_2 \big\| \partial_y \Lambda_\delta^{-2} W_m^\ell f_m \big\|_{L^2(\mathbb{R}_+^2)} \\ + C_2 \big\| \Lambda_\delta^{-2} W_m^\ell f_m \big\|_{L^2(\mathbb{R}_+^2)}. \end{aligned}$$

Now we prove Theorem 1.1 by induction on the estimate of $\phi_m^0 W_m^0 f_m$. The induction procedure is as follows.

Initial hypothesis of the induction. From the hypotheses (1.2) and (1.3) of Theorem 1.1, we have first, in view of (2.1),
(2.7)

$$0 \le m \le N_0+1, \quad \left\|e^{2cy} f_m\right\|_{L^\infty\left([0,T];\ L^2(\mathbb{R}^2_+)\right)} + \sum_{i=1}^{3} \left\|e^{2cy}\partial_y^i f_m\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} < C_0.$$

Hypothesis of the induction. Suppose that there exists $A > C_0+1$ such that, for some $m \ge N_0+1$ and for any $N_0+1 \le k \le m$, we have

$$\partial_y^3 \Lambda^{-1}\phi_k^0 W_k^0 f_k \in L^2([0,T]\times\mathbb{R}^2_+), \tag{2.8}$$

$$\|\phi_k^0 W_k^0 f_k\|_{L^\infty\left([0,T];\ L^2(\mathbb{R}^2_+)\right)} + \sum_{j=1}^{2} \|\partial_y^j \Lambda^{-\frac{2(j-1)}{3}}\phi_k^0 W_k^0 f_k\|_{L^2([0,T]\times\mathbb{R}^2_+)} \le A^{k-5}\left((k-5)!\right)^{3(1+\sigma)}. \tag{2.9}$$

Claim I_{m+1}. We claim that (2.8) and (2.9) are also true for $m+1$. As a result, (2.8) and (2.9) hold for all $k \ge N_0+1$ by induction.

Completeness of the proof for Theorem 1.1. Before proving the above Claim $\mathrm{I}_{\mathrm{m+1}}$, we remark that Theorem 1.1 is just its immediate consequence. Indeed, induction processes imply that for any $m > 1+N_0$, we have for any $0<t<T$,

$$\left\|\phi_m^0 W_m^0 f_m(t)\right\|_{L^2(\mathbb{R}^2_+)} \le A^{m-5}\left((m-5)!\right)^{3(1+\sigma)} \le A^m\left(m!\right)^{3(1+\sigma)};$$

then with (2.2), (2.3), (2.4), and (2.6), we get

$$\begin{aligned}&\forall\ 0<t\le T_1<T\le 1,\\ &t^{3(m-N_0-1)}\left\|e^{\tilde c y}\partial_x^m u\right\|_{L^2(\mathbb{R}^2_+)} \le (T-T_1)^{-3(m-N_0-1)}\left\|\phi_m^0 W_m^0 f_m\right\|_{L^2(\mathbb{R}^2_+)}\end{aligned}$$

yields, for any $m > N_0+1$,

$$\begin{aligned}&\forall\ 0<t\le T_1<T\le 1,\\ &t^{3(m-N_0-1)}\left\|e^{\tilde c y}\partial_x^m u\right\|_{L^2(\mathbb{R}^2_+)} \le (T-T_1)^{-3(m-N_0-1)} A^m\left(m!\right)^{3(1+\sigma)}\\ &\qquad\qquad\qquad\qquad\qquad\le (T-T_1)^{-3m} A^m\left(m!\right)^{3(1+\sigma)}.\end{aligned}$$

As a result, Theorem 1.1 follows if we take $L=(T-T_1)^{-3}A$. □

Now we begin to prove Claim $\mathrm{I}_{\mathrm{m+1}}$, and to do so it is sufficient to prove the following.

Claim $\mathrm{E}_{\mathrm{m},\ell}(0\le\ell\le 3)$. The following properties hold for $0\le\ell\le 3$:

$$\begin{aligned}&\partial_y^3\Lambda^{-1}\phi_m^\ell W_m^\ell f_m \in L^2([0,T]\times\mathbb{R}^2_+),\\ &\|\phi_m^\ell W_m^\ell f_m\|_{L^\infty\left([0,T];\ L^2(\mathbb{R}^2_+)\right)} + \sum_{j=1}^{2}\|\partial_y^j\Lambda^{-\frac{2(j-1)}{3}}\phi_m^\ell W_m^\ell f_m\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\ &\qquad \le A^{m-5+\frac{\ell}{6}}\left((m-5)!\right)^{3(1+\sigma)}(m-4)^{\ell(1+\sigma)}.\end{aligned} \tag{2.10}$$

In fact, Claim $\mathrm{E_{m,3}}$ yields $\partial_y^3\Lambda^{-1}\phi_m^3 W_m^3 f_m \in L^2([0,T]\times\mathbb{R}_+^2)$ and

$$\begin{aligned}\|\phi_m^3 W_m^3 f_m\|_{L^\infty\left([0,T];\ L^2(\mathbb{R}_+^2)\right)} &+ \sum_{j=1}^{2}\|\partial_y^j\Lambda^{-\frac{2(j-1)}{3}}\phi_m^3 W_m^3 f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\ &\leq A^{m-5+\frac12}\left((m-5)!\right)^{3(1+\sigma)}(m-4)^{3(1+\sigma)}\\ &= A^{m-5+\frac12}\left[\left((m+1)-5\right)!\right]^{3(1+\sigma)},\end{aligned}$$

which, along with Lemma 2.2, yields $\partial_y^3\Lambda^{-1}\phi_{m+1}^0 W_{m+1}^0 f_{m+1} \in L^2([0,T]\times\mathbb{R}_+^2)$ and

$$\begin{aligned}\|\phi_{m+1}^0 W_{m+1}^0 f_{m+1}\|_{L^\infty\left([0,T];\ L^2(\mathbb{R}_+^2)\right)} &+ \sum_{j=1}^{2}\|\partial_y^j\Lambda^{-\frac{2(j-1)}{3}}\phi_{m+1}^0 W_{m+1}^0 f_{m+1}\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\ &\leq C_2 A^{m-5+\frac12}\left[\left((m+1)-5\right)!\right]^{3(1+\sigma)},\end{aligned}$$

recalling C_2 is a constant depending only on the numbers σ, c, and the constants C_0, C_* in Theorem 1.1. As a result, if we choose A in such a way that

$$A^{1/2}\geq C_2,$$

then we see (2.9) is also valid for $k=m+1$. Thus the desired Claim $\mathrm{I_{m+1}}$ follows.

Proof of the Claim $E_{m,\ell}$. The rest of this section is devoted to proving Claim $\mathrm{E}_{\mathrm{m},\ell}$ holds for all $0\leq\ell\leq3$, supposing the inductive hypotheses (2.8) and (2.9) hold.

We will prove Claim $\mathrm{E}_{\mathrm{m},\ell}$ by iteration on $0\leq\ell\leq3$. Obviously Claim $\mathrm{E_{m,0}}$ holds, due to the hypothesis of induction (2.8) and (2.9) with $k=m$. Now supposing Claim $\mathrm{E_{m,i}}$ holds for all $0\leq i\leq\ell-1$, i.e., for all $0\leq i\leq\ell-1$, we have

$$\partial_y^3\Lambda^{-1}\phi_m^i W_m^i f_m \in L^2([0,T]\times\mathbb{R}_+^2),$$

$$\begin{aligned}\|\phi_m^i W_m^i f_m\|_{L^\infty\left([0,T];\ L^2(\mathbb{R}_+^2)\right)} &+ \sum_{j=1}^{2}\|\partial_y^j\Lambda^{-\frac{2(j-1)}{3}}\phi_m^i W_m^i f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\ &\leq A^{m-5+\frac{i}{6}}\left((m-5)!\right)^{3(1+\sigma)}(m-4)^{i(1+\sigma)};\end{aligned}\tag{2.11}$$

we will prove in the remaining part Claim $\mathrm{E}_{\mathrm{m},\ell}$ also holds. To do so, we first introduce the mollifier $\Lambda_\delta^{-2}=\Lambda_{\delta,x}^{-2}$ which is the Fourier multiplier with the symbol $\langle\delta\xi\rangle^{-2}$, $0<\delta<1$, and then consider the function $F=\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell f_m$. Under the inductive assumption (2.11), we see F is a classical solution to the following problem (see the detail computation in section 6 and (6.1) fulfilled by f_m):

$$\begin{cases}\left(\partial_t+u\partial_x+v\partial_y-\partial_y^2\right)F=\mathcal{Z}_{m,\ell,\delta},\\ \partial_y F\big|_{y=0}=0,\\ F\big|_{t=0}=0,\end{cases}\tag{2.12}$$

where

$$\begin{aligned}\mathcal{Z}_{m,\ell,\delta}=&\ \Lambda_\delta^{-2}\phi_m^\ell W_m^\ell\mathcal{Z}_m\\ &+\Lambda_\delta^{-2}\left(\partial_t\phi_m^\ell\right)W_m^\ell f_m+\left[u\partial_x+v\partial_y-\partial_y^2,\ \Lambda_\delta^{-2}\phi_m^\ell W_m^\ell\right]f_m\end{aligned}\tag{2.13}$$

with $\mathcal{Z}_m$ given in the appendix (see section 6), that is,

$$\mathcal{Z}_m = -\sum_{j=1}^{m}\binom{m}{j}(\partial_x^j u) f_{m+1-j} - \sum_{j=1}^{m-1}\binom{m}{j}(\partial_x^j v)(\partial_y f_{m-1})$$
$$- \left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]\sum_{j=1}^{m-1}\binom{m}{j}(\partial_x^j v)(\partial_x^{m-j}u) - 2\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right] f_m.$$

The initial value and boundary value in (2.12) are taken in the sense of a trace in Sobolev space, due to the induction hypothesis (2.9) and the facts that $\partial_y\Lambda_\delta^{-2}\phi_m^\ell f_m|_{y=0} = 0$ (see (6.5) in the appendix) and

$$\partial_y\left(e^{2cy}\left(1+\frac{2cy}{(3m+i)\sigma}\right)^{-(3m+i)\sigma/2}(1+cy)^{-1}\right)\bigg|_{y=0} = 0.$$

We will prove an energy estimate for (2.12). For this purpose, let $t \in [0,T]$, and take the $L^2\left([0,t]\times\mathbb{R}_+^2\right)$ inner product with F on both sides of the first equation in (2.12); this gives

$$\mathrm{Re}\ \left(\left(\partial_t + u\partial_x + v\partial_y - \partial_y^2\right)F,\ F\right)_{L^2\left([0,t]\times\mathbb{R}_+^2\right)} = \mathrm{Re}\ \left(\mathcal{Z}_{m,\ell,\delta},\ F\right)_{L^2\left([0,t]\times\mathbb{R}_+^2\right)}.$$

Moreover observing the initial boundary conditions in (2.12) and the facts that $u|_{y=0} = v|_{y=0} = 0$ and $\partial_x u + \partial_y v = 0$, we integrate by parts to obtain

$$\mathrm{Re}\ \left(\left(\partial_t + u\partial_x + v\partial_y - \partial_y^2\right)F,\ F\right)_{L^2\left([0,t]\times\mathbb{R}_+^2\right)} = \frac{1}{2}\|F(t)\|^2_{L^2(\mathbb{R}_+^2)} + \int_0^t \|\partial_y F(t)\|^2_{L^2(\mathbb{R}_+^2)}dt.$$

Thus we infer

$$\|F\|^2_{L^\infty\left([0,T];\ L^2(\mathbb{R}_+^2)\right)} + \|\partial_y F\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} \le 2\left|\left(\mathcal{Z}_{m,\ell,\delta},\ F\right)_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\right|,$$

and thus

$$\begin{aligned}
&\|F\|^2_{L^\infty\left([0,T];\ L^2(\mathbb{R}_+^2)\right)} + \|\partial_y F\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} + \|\partial_y^2\Lambda^{-2/3}F\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
&\le\ 2\left|\left(\mathcal{Z}_{m,\ell,\delta},\ F\right)_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\right| + \|\partial_y^2\Lambda^{-2/3}F\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
&\le\ 2\left\|\phi^{1/2}\mathcal{Z}_{m,\ell,\delta}\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\left\|\phi^{-1/2}F\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} + \|\partial_y^2\Lambda^{-2/3}F\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}.
\end{aligned}\tag{2.14}$$

In order to treat the first term on the right-hand side, we need the following proposition, whose proof is postponed to section 5.

Proposition 2.3. *Under the induction hypothesis* (2.7)–(2.9) *and* (2.11), *there exists a constant* C_3 *such that, using the notation* $F = \Lambda_\delta^{-2}\phi_m^\ell W_m^\ell f_m$ *and* $\tilde{f} = \phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m$ *with* ϕ *defined in* (2.6),

$$\begin{aligned}
&\left\|\phi^{1/2}\mathcal{Z}_{m,\ell,\delta}\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
&\le mC_3\left\|\phi^{-1/2}F\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)} + C_3\left\|\partial_y F\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
&\quad + C_3 A^{m-6}\left((m-5)!\right)^{3(1+\sigma)},
\end{aligned}\tag{2.15}$$

$$\begin{aligned}&\big\|\Lambda^{-1/3}\phi^{1/2}\mathcal{Z}_{m,\ell-1,\delta}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\&\leq mC_3\big\|\Lambda^{-1/3}\phi^{-1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\&\quad+C_3\big\|\partial_y\Lambda^{-1/3}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}+C_3A^{m-6}\left((m-5)!\right)^{3(1+\sigma)},\end{aligned}\tag{2.16}$$

and

$$\begin{aligned}&\big\|\Lambda^{-\frac{2}{3}}\partial_y\phi^{1/2}\mathcal{Z}_{m,\ell-1,\delta}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\&\leq C_3\|\left\langle y\right\rangle^{-\sigma}\Lambda^{1/3}\tilde f\|_{L^2([0,T]\times\mathbb{R}^2_+)}+C_3\big\|\partial_y^2\Lambda^{-2/3}\tilde f\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\&\quad+m\,C_3\left(\big\|\Lambda^{-2/3}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\right)\\&\quad+\big\|\Lambda^{-2/3}\phi^{-1/2}\partial_y\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\&\quad+C_3A^{m-6}\left((m-5)!\right)^{3(1+\sigma)}.\end{aligned}\tag{2.17}$$

The constant C_3 depends only on σ, c, and the constant C_*, but is independent of m and δ.

Now combining (2.15) in the above proposition with (2.14), we have

$$\begin{aligned}&\|F\|^2_{L^\infty\left([0,T];\ L^2(\mathbb{R}^2_+)\right)}+\sum_{j=1}^2\big\|\partial_y^j\Lambda^{-\frac{2(j-1)}{3}}F\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\&\leq 2mC_3\big\|\phi^{-1/2}F\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}+(2C_3)^2\big\|\phi^{-1/2}F\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}+\frac{1}{2}\big\|\partial_yF\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\&\quad+\left(A^{m-6}\left((m-5)!\right)^{3(1+\sigma)}\right)^2+\|\partial_y^2\Lambda^{-2/3}F\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)},\end{aligned}$$

which yields, denoting by $C_4=4C_3+10C_3^2+2$,

$$\begin{aligned}&\|F\|^2_{L^\infty\left([0,T];\ L^2(\mathbb{R}^2_+)\right)}+\sum_{j=1}^2\big\|\partial_y^j\Lambda^{-\frac{2(j-1)}{3}}F\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\&\leq mC_4\big\|\phi^{-1/2}F\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}+2\left(A^{m-6}\left((m-5)!\right)^{3(1+\sigma)}\right)^2\\&\quad+2\|\partial_y^2\Lambda^{-2/3}F\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)},\end{aligned}$$

or equivalently,

$$\begin{aligned}&\|F\|_{L^\infty\left([0,T];\ L^2(\mathbb{R}^2_+)\right)}+\sum_{j=1}^2\big\|\partial_y^j\Lambda^{-\frac{2(j-1)}{3}}F\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\&\leq C_4\left(m^{1/2}\big\|\phi^{-1/2}F\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}+\|\partial_y^2\Lambda^{-2/3}F\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\right)\\&\quad+2A^{m-6}\left((m-5)!\right)^{3(1+\sigma)}.\end{aligned}\tag{2.18}$$

It remains to treat the right terms on the right-hand side. To do so we need to study the subellipticity of the linearized Prandtl equation:

$$\mathcal{P}f=\partial_tf+u\partial_xf+v\partial_yf-\partial_y^2f=h,\quad (t,x,y)\in]0,T[\times\mathbb{R}^2_+,\tag{2.19}$$

where u, v is the solution of Prandtl's equation (1.1) satisfying the conditions (1.2) and (1.3). Then we have the following.

PROPOSITION 2.4. *Let $h, g \in L^2([0,T] \times \mathbb{R}^2_+)$ be given such that $\partial_y h, \partial_y g \in L^2([0,T] \times \mathbb{R}^2_+)$. Suppose that $f \in L^2\left([0,T];\ H^2(\mathbb{R}^2_+)\right)$ with $\partial_y^3 f \in L^2\left([0,T] \times \mathbb{R}^2_+\right)$ is a classical solution to* (2.19) *with the following initial and boundary conditions:*

$$f(0,x,y) = f(T,x,y) = 0, \quad (x,y) \in \mathbb{R}^2_+, \tag{2.20}$$

and

$$\partial_y f(t,x,0) = 0, \quad \partial_t f(t,x,0) = \left(\partial_y^2 f\right)(t,x,0) + g(t,x,0), \quad (t,x) \in]0,T[\times\mathbb{R}. \tag{2.21}$$

Then for any $\varepsilon > 0$ there exists a constant C_ε, depending only on ε, σ, and the constants C_*, such that

$$\begin{aligned}
&\left\| \langle y \rangle^{-\sigma/2} \Lambda^{1/3} f \right\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\|\partial_y^2 \Lambda^{-1/3} f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&\quad \le \varepsilon \left\|\Lambda^{-2/3}\partial_y h\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&\quad + C_\varepsilon \left(\left\|\Lambda^{-1/3} h\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\|\partial_y f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\| f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} \right) \\
&\quad + C_\varepsilon \left(\left\| \langle y \rangle^{-\frac{\sigma}{2}} \partial_y \Lambda^{1/6} f \right\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\| \langle y \rangle^{\sigma} \Lambda^{-1/3} \partial_y g \right\|_{L^2([0,T]\times\mathbb{R}^2_+)} \right).
\end{aligned} \tag{2.22}$$

Moreover,

$$\begin{aligned}
&\left\|\partial_y^3 \Lambda^{-2/3} f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&\le \tilde{C} \Big(\left\| \langle y \rangle^{-\sigma/2} \Lambda^{1/3} f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\|\Lambda^{-2/3}\partial_y h\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&\qquad + \left\|\partial_y f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\| f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} \Big),
\end{aligned}$$

where $\tilde{C}$ is a constant depending only on σ, c, and C_*, C_0 in Theorem 1.1.

We will prove this proposition in section 3. This subellipitic estimate gives a gain of regularity of order $\frac{1}{3}$ with respect to the x variable, so it is sufficient to repeat the same procedure 3 times to get 1 order of regularity.

Continuation of the proof of the Claim $E_{m,\ell}$.

We now use the above subellipticity for the function $f = \tilde{f}$, with $\tilde{f}$ defined in Proposition 2.3, i.e.,

$$f = \phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1} W_m^{\ell-1} f_m = \Lambda_\delta^{-2}\phi^{3(m-N_0-1)+\ell-\frac{1}{2}} W_m^{\ell-1} f_m.$$

Similarly to (2.12), we see f is a classical solution to the following problem:

$$\begin{cases}
\left(\partial_t + u\partial_x + v\partial_y - \partial_y^2\right) f = \phi^{1/2}\mathcal{Z}_{m,\ell-1,\delta} + \left(\partial_t \phi^{1/2}\right)\Lambda_\delta^{-2}\phi_m^{\ell-1} W_m^{\ell-1} f_m, \\
\partial_y f\big|_{y=0} = 0, \\
f\big|_{t=0} = f\big|_{t=T} = 0,
\end{cases}$$

where $\mathcal{Z}_{m,\ell-1,\delta}$ is defined in (2.13). The initial value and boundary value in (2.12) are taken in the sense of a trace in Sobolev space. The validity of Claim $\mathrm{E}_{\mathrm{m},\ell-1}$ due to the inductive assumption (2.11) yields that $\partial_y^3 f \in L^2\left([0,T]\times\mathbb{R}_+^2\right)$. Next we calculate $(\partial_t f-\partial_y^2 f)\big|_{y=0}$. First we have, seeing (6.6) in the appendix,

$$\left(\partial_t f_m-\partial_y^2 f_m\right)|_{y=0}=-2\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]f_m\Big|_{y=0}.$$

Then

$$\begin{aligned}\partial_t f\big|_{y=0}=&\Lambda_\delta^{-2}\left(\partial_t\phi^{3(m-N_0-1)+\ell-\frac12}\right)W_m^{\ell-1}f_m\big|_{y=0}+\Lambda_\delta^{-2}\phi^{3(m-N_0-1)+\ell-\frac12}W_m^{\ell-1}\partial_t f_m\big|_{y=0}\\=&\Lambda_\delta^{-2}\left(\partial_t\phi^{3(m-N_0-1)+\ell-\frac12}\right)W_m^{\ell-1}f_m\big|_{y=0}+\Lambda_\delta^{-2}\phi^{3(m-N_0-1)+\ell-\frac12}W_m^{\ell-1}\partial_y^2 f_m\big|_{y=0}\\&-2\Lambda_\delta^{-2}\phi^{3(m-N_0-1)+\ell-\frac12}W_m^{\ell-1}\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]f_m\Big|_{y=0}\\=&\Lambda_\delta^{-2}\left(\partial_t\phi^{3(m-N_0-1)+\ell-\frac12}\right)\Lambda^{(\ell-1)/3}f_m\big|_{y=0}\\&+\Lambda_\delta^{-2}\phi^{3(m-N_0-1)+\ell-\frac12}W_m^{\ell-1}\partial_y^2 f_m\big|_{y=0}\\&-2\Lambda_\delta^{-2}\phi^{3(m-N_0-1)+\ell-\frac12}\Lambda^{(\ell-1)/3}\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]f_m\Big|_{y=0}.\end{aligned}$$

This, along with the fact that

$$\begin{aligned}&\Lambda_\delta^{-2}\phi^{3(m-N_0-1)+\ell-\frac12}W_m^{\ell-1}\partial_y^2 f_m\big|_{y=0}\\&\quad=\partial_y^2\Lambda_\delta^{-2}\phi^{3(m-N_0-1)+\ell-\frac12}W_m^{\ell-1}f_m\big|_{y=0}-\left[\partial_y^2,\ W_m^{\ell-1}\right]\Lambda_\delta^{-2}\phi^{3(m-N_0-1)+\ell-\frac12}f_m\big|_{y=0}\\&\quad=\partial_y^2 f\big|_{y=0}-\left(\frac{2c^2}{(3m+\ell-1)\sigma}+3c^2\right)\Lambda_\delta^{-2}\phi^{3(m-N_0-1)+\ell-\frac12}\Lambda^{(\ell-1)/3}f_m\big|_{y=0}\end{aligned}$$

due to the fact that $\partial_y\Lambda_\delta^{-2}f_m|_{y=0}=0$ (see (6.5) in the appendix), gives

$$\begin{aligned}&\left(\partial_t f-\partial_y^2 f\right)|_{y=0}\\&\quad=\Lambda_\delta^{-2}\left(\partial_t\phi^{3(m-N_0-1)+\ell-\frac12}\right)\Lambda^{(\ell-1)/3}f_m\big|_{y=0}\\&\qquad-\left(\frac{2c^2}{(3m+\ell-1)\sigma}+3c^2\right)\Lambda_\delta^{-2}\phi^{3(m-N_0-1)+\ell-\frac12}\Lambda^{(\ell-1)/3}f_m\big|_{y=0}\\&\qquad-2\Lambda_\delta^{-2}\phi^{3(m-N_0-1)+\ell-\frac12}\Lambda^{(\ell-1)/3}\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]f_m\Big|_{y=0}\\&\quad\overset{\text{def}}{=}g\big|_{y=0}\end{aligned}$$

with

$$\begin{aligned}g=&\Lambda_\delta^{-2}\left(\partial_t\phi^{3(m-N_0-1)+\ell-\frac12}\right)\Lambda^{(\ell-1)/3}f_m\\&-\left(\frac{2c^2}{(3m+\ell-1)\sigma}+3c^2\right)\Lambda_\delta^{-2}\phi^{3(m-N_0-1)+\ell-\frac12}\Lambda^{(\ell-1)/3}f_m\\&-2\Lambda_\delta^{-2}\phi^{3(m-N_0-1)+\ell-\frac12}\Lambda^{(\ell-1)/3}\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]f_m.\end{aligned}\tag{2.23}$$

Then using Proposition 2.4 for $h=\phi^{1/2}\mathcal{Z}_{m,\ell-1,\delta}+\left(\partial_t\phi^{1/2}\right)\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m$ and the above g, we have

$$
\begin{aligned}
&\left\| \langle y\rangle^{-\sigma/2} \Lambda^{1/3} f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\|\partial_y^2\Lambda^{-1/3} f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&\quad\le \varepsilon\left\|\Lambda^{-2/3}\partial_y h\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&\quad + C_\varepsilon\left(\left\|\Lambda^{-1/3}h\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\|\partial_y f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\|f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}\right) \\
&\quad + C_\varepsilon\left(\left\| \langle y\rangle^{-\frac{\sigma}{2}} \partial_y\Lambda^{1/6} f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\| \langle y\rangle^{\sigma} \Lambda^{-1/3}\partial_y g\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}\right).
\end{aligned}
$$

We claim. for any $\tilde\varepsilon > 0$,

$$
\begin{aligned}
&\varepsilon\left\|\Lambda^{-2/3}\partial_y h\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&\quad + C_\varepsilon\left(\left\|\Lambda^{-1/3}h\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\|\partial_y f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\|f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}\right) \\
&\quad + C_\varepsilon\left(\left\| \langle y\rangle^{-\frac{\sigma}{2}} \partial_y\Lambda^{1/6} f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\| \langle y\rangle^{\sigma} \Lambda^{-1/3}\partial_y g\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}\right) \\
&\quad \le \varepsilon\, C_5\left(\left\| \langle y\rangle^{-\sigma/2} \Lambda^{1/3} f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\|\partial_y^2\Lambda^{-1/3} f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}\right) \\
&\quad + \tilde\varepsilon\, m^{-(1+\sigma)/2}\left(\left\|F\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\|\partial_y F\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}\right) \\
&\quad + C_{\varepsilon,\tilde\varepsilon} m^{(1+\sigma)/2} A^{m-5+\frac{\ell-1}{6}}\left((m-5)!\right)^s m^{(\ell-1)(1+\sigma)},
\end{aligned}
\tag{2.24}
$$

where C_5 is a constant depending only on σ, c, and the constant C_*, but independent of m and δ, and $C_{\varepsilon,\tilde\varepsilon}$ is a constant depending only on $\varepsilon, \tilde\varepsilon, \sigma$, c, and the constant C_*, but independent of m and δ. Recall $F = \Lambda_\delta^{-2}\phi_m^\ell W_m^\ell f_m$. The proof of (2.24) is postponed to the end of this section. Now combining the above inequalities and letting ε be small enough, we infer for any $\tilde\varepsilon > 0$,

$$
\begin{aligned}
&\left\| \langle y\rangle^{-\sigma/2} \Lambda^{1/3} f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\|\partial_y^2\Lambda^{-1/3} f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&\quad \le \tilde\varepsilon\, m^{-(1+\sigma)/2}\left(\left\|F\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\|\partial_y F\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}\right) \\
&\quad + C_{\tilde\varepsilon} m^{(1+\sigma)/2} A^{m-5+\frac{\ell-1}{6}}\left((m-5)!\right)^s m^{(\ell-1)(1+\sigma)}.
\end{aligned}
\tag{2.25}
$$

Now we come back to estimate the terms on the right side of (2.18). To do so we need the following technical lemma, whose proof is presented at the end of section 4.

Lemma 2.5. *Recall* $F = \Lambda_\delta^{-2}\phi_m^\ell W_m^\ell f_m$ *and* $f = \phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m$. *There exists a constant* C_6*, depending only on* σ, c*, and the constant* C_**, but independent of* m *and* δ*, such that*

$$
\begin{aligned}
&\|\phi^{-1/2}F\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\|\partial_y^2\Lambda^{-2/3} F\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&\le C_6\left(m^{\sigma/2}\left\| \langle y\rangle^{-\frac{\sigma}{2}} \Lambda^{1/3} f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\|\partial_y^2\Lambda^{-1/3} f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}\right) \\
&\quad + C_6\left(\|\phi_m^{\ell-1}W_m^{\ell-1}f_m\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\|\partial_y\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\|\partial_y^3\Lambda^{-1}F\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&\le C_6\left\|\partial_y^2\Lambda^{-1/3} f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} + C_6\left\|\partial_y^3\Lambda^{-2/3} f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&\quad + C_6\left(\|\phi_m^{\ell-1}W_m^{\ell-1}f_m\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\|\partial_y\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}\right).
\end{aligned}
\tag{2.26}
$$

End of the proof of the Claim $E_{m,\ell}$. We combine (2.25) and the first estimate in Lemma 2.5, to conclude

$$
\begin{aligned}
&\|\phi^{-1/2}F\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\|\partial_y^2\Lambda^{-2/3}F\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\leq\ & \tilde{\varepsilon}C_6 m^{-1/2}\left(\left\|F\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\|\partial_y F\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}\right)\\
&+C_6C_{\tilde{\varepsilon}}m^{\frac{1}{2}+\sigma}A^{m-5+\frac{\ell-1}{6}}\left((m-5)!\right)^s m^{(\ell-1)(1+\sigma)}\\
&+C_6\left(\|\phi_m^{\ell-1}W_m^{\ell-1}f_m\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\|\partial_y\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}\right)\\
\leq\ & \tilde{\varepsilon}C_6 m^{-1/2}\left(\left\|F\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\|\partial_y F\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}\right)\\
&\ + \left(C_6C_{\tilde{\varepsilon}}+C_6\right)m^{\frac{1}{2}+\sigma}A^{m-5+\frac{\ell-1}{6}}\left((m-5)!\right)^{3(1+\sigma)}(m-4)^{(\ell-1)(1+\sigma)},
\end{aligned}
$$

the last inequality using (2.11). This along with (2.18) yields

$$
\begin{aligned}
&\|F\|_{L^\infty\left([0,T];\ L^2(\mathbb{R}^2_+)\right)} + \sum_{j=1}^{2}\left\|\partial_y^j\Lambda^{-\frac{2(j-1)}{3}}F\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&\leq \tilde{\varepsilon}C_4C_6\left(\left\|F\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + \|\partial_y F\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\right)\\
&\quad +C_4\left(C_6C_{\tilde{\varepsilon}}+C_6\right)m^{1+\sigma}A^{m-5+\frac{\ell-1}{6}}\left((m-5)!\right)^{3(1+\sigma)}(m-4)^{(\ell-1)(1+\sigma)}\\
&\quad +2A^{m-6}\left((m-5)!\right)^{3(1+\sigma)}.
\end{aligned}
$$

Consequently, letting $\tilde{\varepsilon}>0$ be small sufficiently,

$$
\begin{aligned}
&\|F\|_{L^\infty\left([0,T];\ L^2(\mathbb{R}^2_+)\right)} + \sum_{j=1}^{2}\left\|\partial_y^j\Lambda^{-\frac{2(j-1)}{3}}F\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&\leq C_7m^{1+\sigma}A^{m-5+\frac{\ell-1}{6}}\left((m-5)!\right)^{3(1+\sigma)}(m-4)^{(\ell-1)(1+\sigma)} + C_7A^{m-6}\left((m-5)!\right)^{3(1+\sigma)}\\
&\leq C_8(m-4)^{1+\sigma}A^{m-5+\frac{\ell-1}{6}}\left((m-5)!\right)^{3(1+\sigma)}(m-4)^{(\ell-1)(1+\sigma)}\\
&\quad +C_7A^{m-6}\left((m-5)!\right)^{3(1+\sigma)},
\end{aligned}
$$

where C_7, C_8 are two constants depending only on σ, c, and the constants C_0, C_* in Theorem 1.1, but are independent of m and δ. Now we choose A such that

$$A \geq (2C_8+2C_7+1)^6.$$

It then follows that, observing $\ell\geq 1$,

$$
\begin{aligned}
&\|F\|_{L^\infty\left([0,T];\ L^2(\mathbb{R}^2_+)\right)} + \sum_{j=1}^{2}\left\|\partial_y^j\Lambda^{-\frac{2(j-1)}{3}}F\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&\qquad \leq A^{m-5+\frac{\ell}{6}}\left((m-5)!\right)^s(m-4)^{\ell(1+\sigma)}.
\end{aligned}
$$

Observe that the above constant A is independent of δ, and thus letting $\delta\to 0$, we see (2.11) holds for $i=\ell$. It remains to prove that $\partial_y^3\Lambda^{-1}\phi_m^\ell W_m^\ell f_m\in L^2$. The above estimate together with (2.25) gives

$$\left\|\langle y\rangle^{-\sigma/2}\Lambda^{1/3}f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\|\partial_y^2\Lambda^{-1/3}f\right\|_{L^2([0,T]\times\mathbb{R}^2_+)} < C_{m,1}$$

with $C_{m,1}$ a constant depending on m but independent of δ, and thus, using the last estimate in Proposition 2.4 and (2.24),

$$\left\|\partial_y^3\Lambda^{-2/3}f\right\|_{L^2([0,T]\times\mathbb{R}_+^2)} \leq C_{m,2}$$

with $C_{m,2}$ a constant depending on m but independent of δ. As a result, combining with (2.26), we conclude

$$\left\|\partial_y^3\Lambda^{-1}F\right\|_{L^2([0,T]\times\mathbb{R}_+^2)} < C_{m,3}$$

with $C_{m,3}$ a constant depending on m but independent of δ. Thus letting $\delta\to 0$, we see $\partial_y^3\Lambda^{-1}\phi_m^\ell W_m^\ell f_m \in L^2([0,T]\times\mathbb{R}_+^2)$. Thus Claim $\mathrm{E}_{\mathrm{m},\ell}$ holds. This completes the proof of Claim $\mathrm{I}_{\mathrm{m}+1}$, and thus the proof of Theorem 1.1. ∎

We end up this section by the following

Proof of the estimate (2.24). In the proof we use C to denote different constants depending only on σ, c, and the constants C_0, C_* in Theorem 1.1, but is independent of m and δ.

(a) We first estimate $\left\|\Lambda^{-1/3}h\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}$, recalling

$$h=\phi^{1/2}\mathcal{Z}_{m,\ell-1,\delta}+\left(\partial_t\phi^{1/2}\right)\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m.$$

Using interpolation inequality gives, observing that $\left|\partial_t\phi^{1/2}\right|\leq\phi^{-1/2}$,

$$\begin{aligned}
&\left\|\Lambda^{-1/3}\left(\partial_t\phi^{1/2}\right)\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2(\mathbb{R}_x)}\\
\leq\ & m^{-1/2}\phi^{1/2}\left\|\left(\partial_t\phi^{1/2}\right)\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2(\mathbb{R}_x)}\\
&+m^{(\ell+1)/2}\phi^{-(\ell+1)/2}\left\|\Lambda^{-1-\frac{\ell-1}{3}}\left(\partial_t\phi^{1/2}\right)\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2(\mathbb{R}_x)}\\
\leq\ & m^{-1/2}\left\|\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2(\mathbb{R}_x)}\\
&+m^{(\ell+1)/2}\left\|\Lambda^{-1-\frac{\ell-1}{3}}\phi^{-(\ell+2)/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2(\mathbb{R}_x)}\\
\leq\ & m^{-1/2}\left\|\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2(\mathbb{R}_x)}+m^{(\ell+1)/2}\left\|\Lambda^{-1}\Lambda_\delta^{-2}\phi_{m-1}^0W_{m-1}^0f_m\right\|_{L^2(\mathbb{R}_x)},
\end{aligned}$$

the last inequality following from (2.2) which shows $W_i^0, i\geq 1$, is a decreasing sequence of functions as i varies in $\mathbb{N}$, and the fact that

$$\phi^{-(\ell+2)/2}\phi_m^{\ell-1}\leq\phi_{m-1}^0.$$

Moreover, using (2.5) and the inductive assumptions (2.11) and (2.9), we compute, observing $\ell/2+1\leq 3(1+\sigma)$,

$$\begin{aligned}
&m^{-1/2}\left\|\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2(\mathbb{R}_x)}+m^{(\ell+1)/2}\left\|\Lambda^{-1}\Lambda_\delta^{-2}\phi_{m-1}^0W_{m-1}^0f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
\leq\ & m^{-1/2}\left\|\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2(\mathbb{R}_x)}+\tilde{C}_1m^{(\ell+1)/2}\left\|\phi_{m-1}^0W_{m-1}^0f_{m-1}\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
\leq\ & m^{-1/2}A^{m-5+\frac{\ell-1}{6}}\left((m-5)!\right)^{3(1+\sigma)}(m-4)^{(\ell-1)(1+\sigma)}\\
&+\tilde{C}_1m^{(\ell+1)/2}A^{m-6}\left((m-6)!\right)^{3(1+\sigma)}\\
\leq\ & Cm^{-1/2}A^{m-5+\frac{\ell-1}{6}}\left((m-5)!\right)^{3(1+\sigma)}(m-4)^{(\ell-1)(1+\sigma)}.
\end{aligned}$$

Thus we have, combining the above inequalities,

$$\begin{aligned}&\left\|\Lambda^{-1/3}\left(\partial_t\phi^{1/2}\right)\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\ &\leq Cm^{-1/2}A^{m-5+\frac{\ell-1}{6}}\left((m-5)!\right)^{3(1+\sigma)}(m-4)^{(\ell-1)(1+\sigma)}.\end{aligned}\tag{2.27}$$

Similarly, we can show that

$$\begin{aligned}&\left\|\partial_y\Lambda^{-1/3}\left(\partial_t\phi^{1/2}\right)\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\ &\leq Cm^{-1/2}A^{m-5+\frac{\ell-1}{6}}\left((m-5)!\right)^{3(1+\sigma)}(m-4)^{(\ell-1)(1+\sigma)}.\end{aligned}\tag{2.28}$$

Using (2.16) in Proposition 2.3, we have

$$\begin{aligned}&\left\|\Lambda^{-1/3}\phi^{1/2}\mathcal{Z}_{m,\ell-1,\delta}\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\ &\leq mC_3\left\|\Lambda^{-1/3}\phi^{-1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\ &\quad+C_3\left\|\partial_y\Lambda^{-1/3}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\ &\quad+C_3A^{m-6}\left((m-5)!\right)^s,\end{aligned}$$

and, moreover, repeating the arguments as in (2.27) and (2.28), with $\partial_t\phi^{1/2}$ replaced by $\phi^{-1/2}$,

$$\begin{aligned}&mC_3\left\|\Lambda^{-1/3}\phi^{-1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\ &\quad+C_3\left\|\partial_y\Lambda^{-1/3}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\ &\leq Cm^{1/2}A^{m-5+\frac{\ell-1}{6}}\left((m-5)!\right)^{3(1+\sigma)}(m-4)^{(\ell-1)(1+\sigma)},\end{aligned}$$

and thus

$$\begin{aligned}&\left\|\Lambda^{-1/3}\phi^{1/2}\mathcal{Z}_{m,\ell-1,\delta}\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\ &\leq Cm^{1/2}A^{m-5+\frac{\ell-1}{6}}\left((m-5)!\right)^{3(1+\sigma)}(m-4)^{(\ell-1)(1+\sigma)}.\end{aligned}$$

This along with (2.27) yields

$$\left\|\Lambda^{-1/3}h\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\leq Cm^{1/2}A^{m-5+\frac{\ell-1}{6}}\left((m-5)!\right)^{3(1+\sigma)}(m-4)^{(\ell-1)(1+\sigma)}.$$

(b) In this step we treat $\left\|\Lambda^{-2/3}\partial_y h\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}$. It follows from (2.28) that

$$\begin{aligned}&\left\|\Lambda^{-2/3}\partial_y\left(\partial_t\phi^{1/2}\right)\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\ &\leq Cm^{-1/2}A^{m-5+\frac{\ell-1}{6}}\left((m-5)!\right)^{3(1+\sigma)}(m-4)^{(\ell-1)(1+\sigma)}.\end{aligned}$$

On the other hand, by (2.17) we have, recalling $\tilde f=f=\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m$,

$$\begin{aligned}&\left\|\Lambda^{-\frac{2}{3}}\partial_y\phi^{1/2}\mathcal{Z}_{m,\ell-1,\delta}\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\ &\leq C_3\|\langle y\rangle^{-\sigma}\Lambda^{1/3}f\|_{L^2([0,T]\times\mathbb{R}_+^2)}+C_3\left\|\partial_y^2\Lambda^{-2/3}f\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\ &\quad+m\,C_3\left(\left\|\Lambda^{-2/3}\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\right)\\ &\quad+\left\|\Lambda^{-2/3}\phi^{-1/2}\partial_y\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\ &\quad+C_3A^{m-6}\left((m-5)!\right)^s,\end{aligned}$$

and, moreover, similarly to (2.27) and (2.28), we have

$$m\,C_3\left(\left\|\Lambda^{-2/3}\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}+\left\|\Lambda^{-2/3}\phi^{-1/2}\partial_y\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\right)$$
$$\le Cm^{1/2}A^{m-5+\frac{\ell-1}{6}}\left((m-5)!\right)^{3(1+\sigma)}(m-4)^{(\ell-1)(1+\sigma)},$$

since $\left|\partial_t\phi^{1/2}\right|\ge 1$. Combining the above three inequalities gives

$$\left\|\Lambda^{-2/3}\partial_y h\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\le C\left(\|\left\langle y\right\rangle^{-\sigma}\Lambda^{1/3}f\|_{L^2([0,T]\times\mathbb{R}_+^2)}+\left\|\partial_y^2\Lambda^{-2/3}f\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\right)$$
$$+Cm^{1/2}A^{m-5+\frac{\ell-1}{6}}\left((m-5)!\right)^{3(1+\sigma)}(m-4)^{(\ell-1)(1+\sigma)}.$$

(c) It follows from the inductive assumption (2.11) that, observing $\phi^{1/2}\le 1$,

$$\sum_{j=0}^{1}\left\|\partial_y^j f\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\le\left\|\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}+\left\|\partial_y\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}$$
$$\le A^{m-5+\frac{\ell-1}{6}}\left((m-5)!\right)^{3(1+\sigma)}(m-4)^{(\ell-1)(1+\sigma)}.$$

Now we estimate $\|\left\langle y\right\rangle^{\sigma}\Lambda^{-1/3}\partial_y g\|_{L^2([0,T]\times\mathbb{R}_+^2)}$, with g as defined in (2.23). It is quite similar to step (a). For instance,

$$\|\left\langle y\right\rangle^{\sigma}\Lambda^{-1/3}\partial_y\Lambda_\delta^{-2}\left(\partial_t\phi^{3(m-N_0-1)+\ell-\frac{1}{2}}\right)\Lambda^{(\ell-1)/3}f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)}$$
$$\le\left\|\Lambda^{-1/3}\phi^{-1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}\partial_y f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}$$
$$\le C\left\|\Lambda^{-1/3}\phi^{-1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}$$
$$+C\left\|\partial_y\Lambda^{-1/3}\phi^{-1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}.$$

Then, similarly to (2.27) and (2.28), we conclude

$$\|\left\langle y\right\rangle^{\sigma}\Lambda^{-1/3}\partial_y\Lambda_\delta^{-2}\left(\partial_t\phi^{3(m-N_0-1)+\ell-\frac{1}{2}}\right)\Lambda^{(\ell-1)/3}f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)}$$
$$\le Cm^{-1/2}A^{m-5+\frac{\ell-1}{6}}\left((m-5)!\right)^{3(1+\sigma)}(m-4)^{(\ell-1)(1+\sigma)}.$$

The other terms in (2.23) can be estimated similarly, and a classical commutator estimate (see Lemma 3.1 in the following section) will be used for treatment of the third term in (2.23). Thus we conclude

$$\|\left\langle y\right\rangle^{\sigma}\Lambda^{-1/3}\partial_y g\|_{L^2([0,T]\times\mathbb{R}_+^2)}\le CA^{m-5+\frac{\ell-1}{6}}\left((m-5)!\right)^{3(1+\sigma)}(m-4)^{(\ell-1)(1+\sigma)}.$$

(d) It remains to estimate $\|\left\langle y\right\rangle^{-\sigma/2}\partial_y\Lambda^{1/6}f\|_{L^2([0,T]\times\mathbb{R}_+^2)}$, and we have

$$\|\left\langle y\right\rangle^{-\sigma/2}\partial_y\Lambda^{1/6}f\|^2_{L^2([0,T]\times\mathbb{R}_+^2)}$$
$$=\|\left\langle y\right\rangle^{-\sigma/2}\partial_y\Lambda^{1/6}\Lambda_\delta^{-2}\phi^{3(m-N_0-1)+\ell-\frac{1}{2}}W_m^{\ell-1}f_m\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}$$
$$=\left(\left\langle y\right\rangle^{-\sigma}\partial_y\Lambda^{1/3}\Lambda_\delta^{-2}\phi^{3(m-N_0-1)+\ell}W_m^{\ell-1}f_m,\partial_y\Lambda_\delta^{-2}\phi^{3(m-N_0-1)+\ell-1}W_m^{\ell-1}f_m\right)_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}$$
$$\le\|\left\langle y\right\rangle^{-\sigma}\partial_y\Lambda^{1/3}\Lambda_\delta^{-2}\phi_m^{\ell}W_m^{\ell-1}f_m\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\left\|\partial_y\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}$$
$$\le C\left(\left\|\partial_y\Lambda_\delta^{-2}\phi_m^{\ell}W_m^{\ell}f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\right)$$
$$+\left\|\Lambda_\delta^{-2}\phi_m^{\ell}W_m^{\ell}f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\left\|\partial_y\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)},$$

the last inequality following from the third estimate in Lemma 2.2. This, along with the inductive assumption (2.11) implies, for any $\tilde{\varepsilon} > 0$,

$$\begin{aligned} &\left\| \langle y \rangle^{-\sigma/2} \partial_y \Lambda^{1/3} f \right\|_{L^2([0,T]\times\mathbb{R}^2_+)} \\ &\leq \tilde{\varepsilon} m^{-(1+\sigma)/2} \Big(\left\| F \right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + \left\| \partial_y F \right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \Big) \\ &\quad + C_{\tilde{\varepsilon}} m^{(1+\sigma)/2} A^{m-5+\frac{\ell-1}{6}} \left((m-5)! \right)^s m^{(\ell-1)(1+\sigma)}, \end{aligned}$$

recalling $F = \Lambda_\delta^{-2} \phi_m^\ell W_m^\ell f_m$.

Now combining the estimates in the above steps (a)–(d), we obtain the desired (2.24). ◻

3. Subelliptic estimate. In this section we prove Proposition 2.4. First we state the following estimates on commutators (see Lemma 3.1) which will be used frequently. Throughout the paper we use $\big[Q_1,\ Q_2\big]$ to denote the commutator between two operators Q_1 and Q_2, which is defined by

$$\big[Q_1,\ Q_2\big] = Q_1 Q_2 - Q_2 Q_1 = -\big[Q_2,\ Q_1\big].$$

We have

$$\big[Q_1,\ Q_2 Q_3\big] = Q_2 \big[Q_1,\ Q_3\big] + \big[Q_1,\ Q_2\big] Q_3. \tag{3.1}$$

Lemma 3.1. *Denote by* $\big[\alpha\big]$ *the largest integer less than or equal to* $\alpha \geq 0$. *For any* $\tau \in \mathbb{R}$ *and* $a \in C_b^{[|\tau|]+1}(\mathbb{R}^2_+)$, *the space of functions such that all their derivatives up to the order of* $[|\tau|]+1$ *are continuous and bounded, there exists* $C > 0$ *such that for suitable function* f *and any* $0 < \delta < 1$,

$$\| [a, \Lambda^\tau \Lambda_\delta^{-2}] f \|_{L^2(\mathbb{R}^2_+)} \leq C \| \Lambda^{\tau-1} \Lambda_\delta^{-2} f \|_{L^2(\mathbb{R}^2_+)}$$

and

$$\| [a\partial_x, \Lambda^\tau \Lambda_\delta^{-2}] f \|_{L^2(\mathbb{R}^2_+)} \leq C \| \Lambda^{\tau} \Lambda_\delta^{-2} f \|_{L^2(\mathbb{R}^2_+)}.$$

The constant C *depends only on* τ *and* $\|a\|_{C_b^{[|\tau|]+1}(\mathbb{R}^2_+)}$.

Since $\Lambda^\tau \Lambda_\delta^{-2}$ is only a Fourier multiplier of the x variable, we can prove the above lemma by direct calculus or pseudodifferential computation; cf. [16, 19]. In this section, we use the above lemma with $a = u$ or $a = v$ and $\tau = -1/3, -2/3$, so that with hypothesis (1.3), the constant in Lemma 3.1 depends only on the constant C_0 in Theorem 1.1.

Proof of Proposition 2.4. Taking the operator $\Lambda^{-2/3}$ on both sides of (2.19), we see the function $\Lambda^{-2/3} f$ satisfies the following equation in $]0, T[\ \times \mathbb{R}^2_+$:

$$\begin{aligned} &\partial_t \Lambda^{-2/3} f + u \partial_x \Lambda^{-2/3} f + v \partial_y \Lambda^{-2/3} f - \partial_y^2 \Lambda^{-2/3} f \\ &= \Lambda^{-2/3} h + \big[u\partial_x + v\partial_y,\ \Lambda^{-2/3} \big] f, \end{aligned} \tag{3.2}$$

and that

$$\Lambda^{-2/3} f\big|_{t=0} = \Lambda^{-2/3} f\big|_{t=T} = 0, \quad \partial_y \Lambda^{-2/3} f\big|_{y=0} = 0 \tag{3.3}$$

due to (2.20) and (2.21), since $\Lambda^{-2/3}$ is an operator acting only on the x variable. Recall $\left[u\partial_x + v\partial_y,\ \Lambda^{-2/3}\right]$ stands for the commutator between $u\partial_x + v\partial_y$ and $\Lambda^{-2/3}$.

Step (1). We will show in this step that

$$\begin{aligned}
&\left\| (\partial_y u)^{1/2}\, \partial_x \Lambda^{-2/3} f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&\le 2 \left| \mathrm{Re}\ \left(\partial_t \Lambda^{-2/3} f,\ \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0,T]\times\mathbb{R}^2_+)} \right| + \left\| \partial_y^2 \Lambda^{-1/3} f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&\quad + C \left(\left\| \Lambda^{-1/3} h \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\| \partial_y f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\| f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \right).
\end{aligned} \tag{3.4}$$

To do so, we take the $L^2([0,T]\times\mathbb{R}^2_+)$ inner product with the function $\partial_y \partial_x \Lambda^{-2/3} f \in L^2([0,T]\times\mathbb{R}^2_+)$ on both sides of (3.2), and then consider the real parts; this gives

$$\begin{aligned}
&- \mathrm{Re}\ \left(u\partial_x \Lambda^{-2/3} f,\ \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&= \mathrm{Re}\ \left(\partial_t \Lambda^{-2/3} f,\ \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&\quad - \mathrm{Re}\ \left(\partial_y^2 \Lambda^{-2/3} f,\ \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&\quad + \mathrm{Re}\ \left(v\partial_y \Lambda^{-2/3} f,\ \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&\quad - \mathrm{Re}\ \left(\Lambda^{-2/3} h,\ \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&\quad - \mathrm{Re}\ \left(\left[u\partial_x + v\partial_y,\ \Lambda^{-2/3}\right] f,\ \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0,T]\times\mathbb{R}^2_+)}.
\end{aligned} \tag{3.5}$$

We will treat the terms on both sides. For the term on the left-hand side we integrate by parts to obtain, here we use $u\big|_{y=0} = 0$,

$$\begin{aligned}
&-\mathrm{Re}\ \left(u\partial_x \Lambda^{-2/3} f,\ \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&= -\frac{1}{2} \left\{ \left(u\partial_x \Lambda^{-2/3} f,\ \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0,T]\times\mathbb{R}^2_+)} \right. \\
&\quad \left. + \left(\partial_y \partial_x \Lambda^{-2/3} f,\ u\partial_x \Lambda^{-2/3} f \right)_{L^2([0,T]\times\mathbb{R}^2_+)} \right\} \\
&= \frac{1}{2} \left\| (\partial_y u)^{1/2}\, \partial_x \Lambda^{-2/3} f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}.
\end{aligned}$$

Next we estimate the terms on the right-hand side and have, by Cauchy–Schwarz's inequality,

$$\begin{aligned}
\left| -\mathrm{Re}\ \left(\partial_y^2 \Lambda^{-2/3} f,\ \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0,T]\times\mathbb{R}^2_+)} \right| &\le \frac{1}{2} \left\| \partial_y^2 \Lambda^{-1/3} f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&\quad + \frac{1}{2} \left\| \partial_y f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}, \\
\left| \mathrm{Re}\ \left(\Lambda^{-2/3} h,\ \partial_y \partial_x \Lambda^{-2/3} f \right)_{L^2([0,T]\times\mathbb{R}^2_+)} \right| &\le \left\| \Lambda^{-1/3} h \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&\quad + \left\| \partial_y f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}
\end{aligned}$$

and

$$\begin{aligned}
&\left|-\mathrm{Re}\ \Big(v\partial_y\Lambda^{-2/3}f,\ \partial_y\partial_x\Lambda^{-2/3}f\Big)_{L^2([0,T]\times\mathbb{R}^2_+)}\right|\\
&\leq\left|\Big(\partial_y f,\ \big[\Lambda^{-2/3},\ v\big]\partial_y\partial_x\Lambda^{-2/3}f\Big)_{L^2([0,T]\times\mathbb{R}^2_+)}\right|\\
&\quad+\left|\Big(v\partial_y f,\ \Lambda^{-2/3}\partial_y\partial_x\Lambda^{-2/3}f\Big)_{L^2([0,T]\times\mathbb{R}^2_+)}\right|\\
&\leq C\big\|\partial_y f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)},
\end{aligned}$$

the last inequality using Lemma 3.1. Finally

$$\begin{aligned}
&\left|-\mathrm{Re}\ \Big(\big[u\partial_x+v\partial_y,\ \Lambda^{-2/3}\big]f,\ \partial_y\partial_x\Lambda^{-2/3}f\Big)_{L^2([0,T]\times\mathbb{R}^2_+)}\right|\\
&\leq\big\|\Lambda^{1/3}\big[u\partial_x+v\partial_y,\ \Lambda^{-2/3}\big]f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|\partial_y f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\leq 2\Big(\big\|\big[u\partial_x+v\partial_y,\ \Lambda^{1/3}\Lambda^{-2/3}\big]f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\Big)\\
&\quad+\big\|\big[u\partial_x+v\partial_y,\ \Lambda^{1/3}\big]\Lambda^{-2/3}f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\quad+\big\|\partial_y f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\leq C\Big(\big\|f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|\partial_y f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\Big),
\end{aligned}$$

These inequalities, together with (3.5), yield the desired (3.4).

Step (2). In this step we will estimate the second term on the right-hand side of (3.4) and show that for any $\varepsilon>0$,

$$\begin{aligned}
&\big\|\partial_y^2\Lambda^{-1/3}f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&\leq\varepsilon\ \|\left(\partial_y u\right)^{1/2}\partial_x\Lambda^{-2/3}f\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&\quad+C_\varepsilon\Big(\big\|\partial_y f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}+\big\|\Lambda^{-1/3}h\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\Big)
\end{aligned}\tag{3.6}$$

with C_ε a constant depending on ε. We see that the function $\Lambda^{-1/3}f$ satisfies the equation in $]0,T[\ \times\mathbb{R}^2_+$,

$$\begin{aligned}
&\partial_t\Lambda^{-1/3}f+\left(u\partial_x+v\partial_y\right)\Lambda^{-1/3}f-\partial_y^2\Lambda^{-1/3}f\\
&=\Lambda^{-1/3}h+\big[u\partial_x+v\partial_y,\ \Lambda^{-1/3}\big]f,
\end{aligned}\tag{3.7}$$

with the boundary condition

$$\Lambda^{-1/3}f\big|_{t=0}=\Lambda^{-1/3}f\big|_{t=T}=0,\quad \partial_y\Lambda^{-1/3}f\big|_{y=0}=0.\tag{3.8}$$

We take the $L^2([0,T]\times\mathbb{R}^2_+)$ inner product with function $-\partial_y^2\Lambda^{-1/3}f\in L^2([0,T]\times\mathbb{R}^2_+)$ on both sides of (3.7), and then consider the real parts; this gives

$$\big\|\partial_y^2\Lambda^{-1/3}f\big\|^2_{L^2(\mathbb{R}^3_+)}\leq\sum_{p=1}^4 J_p,\tag{3.9}$$

where

$$J_1 = \left|\text{Re } \Big(\partial_t \Lambda^{-1/3} f,\ \partial_y^2 \Lambda^{-1/3} f\Big)_{L^2([0,T]\times\mathbb{R}^2_+)}\right|,$$
$$J_2 = \left|\text{Re } \Big(\left(u\partial_x + v\partial_y\right)\Lambda^{-1/3} f,\ \partial_y^2 \Lambda^{-1/3} f\Big)_{[0,T]\times L^2([0,T]\times\mathbb{R}^2_+)}\right|,$$
$$J_3 = \left|\text{Re } \Big(\Lambda^{-1/3} h,\ \partial_y^2 \Lambda^{-1/3} f\Big)_{L^2([0,T]\times\mathbb{R}^2_+)}\right|,$$
$$J_4 = \left|\text{Re } \Big(\big[u\partial_x + v\partial_y,\ \Lambda^{-1/3}\big] f,\ \partial_y^2 \Lambda^{-1/3} f\Big)_{L^2([0,T]\times\mathbb{R}^2_+)}\right|.$$

Integrating by parts and observing the condition (3.8), we see

$$\Big(\partial_t \Lambda^{-1/3} f,\ \partial_y^2 \Lambda^{-1/3} f\Big)_{L^2([0,T]\times\mathbb{R}^2_+)} = -\Big(\partial_t \partial_y \Lambda^{-1/3} f,\ \partial_y \Lambda^{-1/3} f\Big)_{L^2([0,T]\times\mathbb{R}^2_+)},$$

which along with the fact

$$\text{Re } \Big(\partial_t \partial_y \Lambda^{-1/3} f,\ \partial_y \Lambda^{-1/3} f\Big)_{L^2([0,T]\times\mathbb{R}^2_+)} = 0$$

due to (3.8), implies

$$J_1 = \left|\text{Re } \Big(\partial_t \Lambda^{-1/3} f,\ \partial_y^2 \Lambda^{-1/3} f\Big)_{L^2([0,T]\times\mathbb{R}^2_+)}\right| = 0.$$

About J_2, we integrate by parts again and observe the boundary condititon (3.8) to compute

$$\begin{aligned}
&\text{Re}\Big(u\partial_x \Lambda^{-1/3} f,\ \partial_y^2 \Lambda^{-1/3} f\Big)_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&= -\text{Re}\Big(u\partial_x \Lambda^{-1/3}\partial_y f,\ \Lambda^{-1/3}\partial_y f\Big)_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&\quad -\text{Re } \Big((\partial_y u)\partial_x \Lambda^{-1/3} f,\ \Lambda^{-1/3}\partial_y f\Big)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&= \frac{1}{2}\Big(\left(\partial_x u\right)\Lambda^{-1/3}\partial_y f,\ \Lambda^{-1/3}\partial_y f\Big)_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&\quad -\text{Re } \Big((\partial_y u)\partial_x \Lambda^{-1/3} f,\ \Lambda^{-1/3}\partial_y f\Big)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}.
\end{aligned}$$

This gives

$$\begin{aligned}
&\left|\text{Re}\Big(u\partial_x \Lambda^{-1/3} f,\ \partial_y^2 \Lambda^{-1/3} f\Big)_{L^2([0,T]\times\mathbb{R}^2_+)}\right| \\
&\le \|\Lambda^{-1/3}\left(\partial_y u\right)\partial_x \Lambda^{-1/3} f\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\|\partial_y f\|_{L^2([0,T]\times\mathbb{R}^2_+)} + C\|\partial_y f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&\le \Big(\|\left(\partial_y u\right)\Lambda^{-1/3}\partial_x \Lambda^{-1/3} f\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + \|\big[\Lambda^{-1/3},\ \partial_y u\big]\partial_x \Lambda^{-1/3} f\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\Big) \\
&\times \|\partial_y f\|_{L^2([0,T]\times\mathbb{R}^2_+)} + C\|\partial_y f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&\le C\Big(\|\left(\partial_y u\right)^{1/2}\partial_x \Lambda^{-2/3} f\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + \|f\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\Big) \\
&\quad \|\partial_y f\|_{L^2([0,T]\times\mathbb{R}^2_+)} + C\big\|\partial_y f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&\le\ \varepsilon\ \|\left(\partial_y u\right)^{1/2}\partial_x \Lambda^{-2/3} f\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + C_\varepsilon\Big(\big\|\partial_y f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \big\|f\big\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\Big).
\end{aligned}$$

Moreover, integrating by parts, we obtain

$$
\begin{aligned}
&\left|\mathrm{Re}\left(v\partial_y\Lambda^{-1/3}f,\ \partial_y^2\Lambda^{-1/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\right| \\
&= \frac{1}{2}\left|\left(\left(\partial_y v\right)\partial_y\Lambda^{-1/3}f,\ \partial_y\Lambda^{-1/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\right| \\
&\le\ C\left\|\partial_y f\right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
J_2 \le&\ \ \varepsilon\ \|\left(\partial_y u\right)^{1/2}\partial_x\Lambda^{-2/3}f\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&+ C_\varepsilon\left(\left\|\partial_y f\right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\|f\right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\right).
\end{aligned}
\tag{3.10}
$$

It remains to estimate J_3 and J_4. Let $\tilde\varepsilon > 0$ be an arbitrarily small number. Cauchy–Schwarz's inequality gives

$$
\begin{aligned}
J_3 &= \left|\mathrm{Re}\ \left(\Lambda^{-1/3}h,\ \partial_y^2\Lambda^{-1/3}f\right)_{L^2([0,T]\times\mathbb{R}^2_+)}\right| \\
&\le \tilde\varepsilon\ \left\|\partial_y^2\Lambda^{-1/3}f\right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + C_{\tilde\epsilon}\left\|\Lambda^{-1/3}h\right\|^2_{L^2(\mathbb{R}^3_+)},
\end{aligned}
$$

and for J_4, Lemma 3.1 implies

$$
\begin{aligned}
J_4 &= \left|\mathrm{Re}\ \left(\left[u\partial_x + v\partial_y,\ \Lambda^{-1/3}\right]f,\ \partial_y^2\Lambda^{-1/3}f\right)_{L^2(\mathbb{R}^2_+)}\right| \\
&\le \tilde\varepsilon\ \left\|\partial_y^2\Lambda^{-1/3}f\right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + C_{\tilde\epsilon}\left(\left\|f\right\|^2_{L^2(\mathbb{R}^3_+)} + \left\|\partial_y f\right\|^2_{L^2(\mathbb{R}^3_+)}\right),
\end{aligned}
$$

where $\tilde C_\varepsilon$ is constant depending on $\tilde\varepsilon$. Now the above two estimates for J_3 and J_4, along with (3.9)–(3.10), gives

$$
\begin{aligned}
\left\|\partial_y^2\Lambda^{-1/3}f\right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \le&\ \tilde\varepsilon\ \left\|\partial_y^2\Lambda^{-1/3}f\right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&+\varepsilon\ \|\left(\partial_y u\right)^{1/2}\partial_x\Lambda^{-2/3}f\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&+C_\varepsilon\left(\left\|\partial_y f\right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\|f\right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\right) \\
&+C_{\tilde\epsilon}\left(\left\|\Lambda^{-1/3}h\right\|^2_{L^2(\mathbb{R}^3_+)} + \left\|f\right\|^2_{L^2(\mathbb{R}^3_+)} + \left\|\partial_y f\right\|^2_{L^2(\mathbb{R}^3_+)}\right),
\end{aligned}
$$

and thus, letting $\tilde\varepsilon$ be sufficiently small,

$$
\begin{aligned}
&\left\|\partial_y^2\Lambda^{-1/3}f\right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&\le \varepsilon\ \|\left(\partial_y u\right)^{1/2}\partial_x\Lambda^{-2/3}f\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&\quad +C_\varepsilon\left(\left\|\partial_y f\right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\|f\right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\|\Lambda^{-1/3}h\right\|^2_{L^2(\mathbb{R}^3_+)}\right).
\end{aligned}
$$

This is just the desired estimate (3.6).

Combining the estimates (3.4) and (3.6), we obtain, choosing ε sufficiently small,

$$\begin{aligned}
&\left\| (\partial_y u)^{1/2}\, \partial_x \Lambda^{-2/3} f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\| \partial_y^2 \Lambda^{-1/3} f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&\le C \left| \mathrm{Re}\, \left(\partial_t \Lambda^{-2/3} f,\ \partial_y\partial_x \Lambda^{-2/3} f \right)_{L^2([0,T]\times\mathbb{R}^2_+)} \right| \\
&\quad + C \left(\left\| \Lambda^{-1/3} h \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\| \partial_y f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\| f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \right).
\end{aligned} \tag{3.11}$$

Step (3). It remains to treat the first term on the right-hand side of (3.11). In this step we will prove that, for any $\varepsilon_1 > 0$,

$$\begin{aligned}
&\left| \mathrm{Re}\, \left(\partial_t \Lambda^{-2/3} f,\ \partial_y\partial_x \Lambda^{-2/3} f \right)_{L^2([0,T]\times\mathbb{R}^2_+)} \right| \\
&\le\ \varepsilon_1 \int_0^T \int_{\mathbb{R}} \left| \left(\partial_y^2 \Lambda^{-1/2} f \right)(t,x,0) \right|^2 dxdt \\
&\quad + C_{\varepsilon_1} \left\| \langle y \rangle\, \Lambda^{-1/3} \partial_y g \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&\quad + \varepsilon_1^{-1} C \left(\left\| \langle y \rangle^{-\sigma/2}\, \Lambda^{1/6} f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\| \langle y \rangle^{-\sigma/2}\, \partial_y \Lambda^{1/6} f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \right).
\end{aligned} \tag{3.12}$$

For this purpose we integrate by parts again and observe the boundary condition (3.3), to compute

$$\begin{aligned}
&\left(\partial_t \Lambda^{-2/3} f,\ \partial_y\partial_x \Lambda^{-2/3} f \right)_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&= - \left(\Lambda^{-2/3} f,\ \partial_t\partial_y\partial_x \Lambda^{-2/3} f \right)_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&= \left(\partial_x \Lambda^{-2/3} f,\ \partial_t\partial_y \Lambda^{-2/3} f \right)_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&= - \left(\partial_y\partial_x \Lambda^{-2/3} f,\ \partial_t \Lambda^{-2/3} f \right)_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&\quad + \int_0^T \int_{\mathbb{R}} \left(\partial_t \Lambda^{-2/3} f(t,x,0) \right) \left(\partial_x \Lambda^{-2/3} f(t,x,0) \right) dxdt,
\end{aligned}$$

which, along with the fact that

$$\begin{aligned}
&2\ \mathrm{Re}\, \left(\partial_t \Lambda^{-2/3} f,\ \partial_y\partial_x \Lambda^{-2/3} f \right)_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&= \left(\partial_t \Lambda^{-2/3} f,\ \partial_y\partial_x \Lambda^{-2/3} f \right)_{L^2([0,T]\times\mathbb{R}^2_+)} + \left(\partial_y\partial_x \Lambda^{-2/3} f,\ \partial_t \Lambda^{-2/3} f \right)_{L^2([0,T]\times\mathbb{R}^2_+)},
\end{aligned}$$

yields, for any $\varepsilon_1 > 0$,

$$\begin{aligned}
&\left| \mathrm{Re}\, \left(\partial_t \Lambda^{-2/3} f,\ \partial_y\partial_x \Lambda^{-2/3} f \right)_{L^2([0,T]\times\mathbb{R}^2_+)} \right| \\
&= \frac{1}{2} \left| \int_0^T \int_{\mathbb{R}} \left(\partial_t \Lambda^{-2/3} f(t,x,0) \right) \left(\partial_x \Lambda^{-2/3} f(t,x,0) \right) dxdt \right| \\
&= \frac{1}{2} \left| \int_0^T \int_{\mathbb{R}} \left(\Lambda^{1/6} \partial_t \Lambda^{-2/3} f(t,x,0) \right) \left(\Lambda^{-1/6} \partial_x \Lambda^{-2/3} f(t,x,0) \right) dxdt \right| \\
&\le \varepsilon_1 \int_0^T \int_{\mathbb{R}} \left(\partial_t \Lambda^{-1/2} f(t,x,0) \right)^2 dxdt + \varepsilon_1^{-1} \int_0^T \int_{\mathbb{R}} \left(\Lambda^{1/6} f(t,x,0) \right)^2 dxdt.
\end{aligned} \tag{3.13}$$

Moreover, observing

$$\Lambda^{1/6} f(t,x,0) = \left(\langle y\rangle^{-\sigma/2} \Lambda^{1/6} f \right)(t,x,0),$$

it then follows from the Sobolev inequality that

$$\begin{aligned}\left|\Lambda^{1/6} f(t,x,0)\right|^2 &\leq C\left(\| \langle y\rangle^{-\sigma/2} \Lambda^{1/6} f\|^2_{L^2(\mathbb{R}_+)} + \|\partial_y \langle y\rangle^{-\sigma/2} \Lambda^{1/6} f\|^2_{L^2(\mathbb{R}_+)}\right)\\ &\leq C\left(\| \langle y\rangle^{-\sigma/2} \Lambda^{1/6} f\|^2_{L^2(\mathbb{R}_+)} + \| \langle y\rangle^{-\sigma/2} \partial_y\Lambda^{1/6} f\|^2_{L^2(\mathbb{R}_+)}\right)\end{aligned}$$

with C a constant independent of t, x. And thus

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}} \left(\Lambda^{1/6} f(t,x,0)\right)^2 dxdt\\ &\leq C\left(\| \langle y\rangle^{-\sigma/2} \Lambda^{1/6} f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \|\partial_y \langle y\rangle^{-\sigma/2} \Lambda^{1/6} f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\right)\\ &\leq C\left(\| \langle y\rangle^{-\sigma/2} \Lambda^{1/6} f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \| \langle y\rangle^{-\sigma/2} \Lambda^{1/6}\partial_y f\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\right).\end{aligned} \tag{3.14}$$

Using the fact that

$$\partial_t \Lambda^{-1/2} f(t,x,0) = \left(\partial_y^2 \Lambda^{-1/2} f\right)(t,x,0) + \Lambda^{-1/2} g(t,x,0)$$

due to assumption (2.21), we conclude

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}} \left(\partial_t \Lambda^{-1/2} f(t,x,0)\right)^2 dxdt\\ &\leq \int_0^T \int_{\mathbb{R}} \left|\left(\partial_y^2 \Lambda^{-1/2} f\right)(t,x,0)\right|^2 dxdt + \int_0^T \int_{\mathbb{R}} \left|\Lambda^{-1/2} g(t,x,0)\right|^2 dxdt.\end{aligned}$$

Moreover, observe

$$\begin{aligned}\left|\Lambda^{-1/2} g(t,x,0)\right| &= \left| -\int_0^{+\infty} \partial_{\tilde y} \Lambda^{-1/2} g(t,x,\tilde y) d\tilde y\right|\\ &\leq \left(\int_0^{+\infty} \langle \tilde y\rangle^{-2\sigma} d\tilde y\right)^{1/2} \left(\int_0^{+\infty} \langle \tilde y\rangle^{2\sigma} \left|\Lambda^{-1/2}\partial_{\tilde y} g(t,x,\tilde y)\right|^2 d\tilde y\right)^{1/2},\end{aligned}$$

which implies

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}} \left|\Lambda^{-1/2} g(t,x,0)\right|^2 dxdt \leq C\| \langle y\rangle^{\sigma} \Lambda^{-1/2}\partial_y g\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}\\ &\leq C\| \langle y\rangle^{\sigma} \Lambda^{-1/3}\partial_y g\|^2_{L^2([0,T]\times\mathbb{R}^2_+)},\end{aligned}$$

and thus

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}} \left(\partial_t \Lambda^{-1/2} f(t,x,0)\right)^2 dxdt\\ &\leq \int_0^T \int_{\mathbb{R}} \left|\left(\partial_y^2 \Lambda^{-1/2} f\right)(t,x,0)\right|^2 dxdt + C\| \langle y\rangle^{\sigma} \Lambda^{-1/3}\partial_y g\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}.\end{aligned}$$

This, along with (3.13) and (3.14), yields the desired (3.12).

Step (4). Combining (3.11) and (3.12), we have, for any $\varepsilon_1 > 0$,

$$
\begin{aligned}
&\left\| (\partial_y u)^{1/2}\, \partial_x \Lambda^{-2/3} f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\| \partial_y^2 \Lambda^{-1/3} f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \\
\leq{}& \varepsilon_1 \int_0^T \int_{\mathbb{R}} \left| \left(\partial_y^2 \Lambda^{-1/2} f \right)(t,x,0) \right|^2 dx dt + C_{\varepsilon_1} \left\| \langle y \rangle^{\sigma} \Lambda^{-1/3} \partial_y g \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&+ \varepsilon_1^{-1} C \left(\left\| \langle y \rangle^{-\sigma/2} \Lambda^{1/6} f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\| \langle y \rangle^{-\sigma/2} \partial_y \Lambda^{1/6} f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \right) \\
&+ C \left(\left\| \partial_y f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\| f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\| \Lambda^{-1/3} h \right\|^2_{L^2(\mathbb{R}^3_+)} \right).
\end{aligned}
$$

Moreover, we use the monotonicity condition and interpolation inequality to get, for any $\varepsilon_2 > 0$,

$$
\begin{aligned}
&\left\| \langle y \rangle^{-\sigma/2} \Lambda^{1/6} f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \\
\leq{}& \varepsilon_2 \left\| \langle y \rangle^{-\sigma/2} \Lambda^{1/3} f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \varepsilon_2^{-1} \left\| \langle y \rangle^{-\sigma/2} f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \\
\leq{}& \varepsilon_2 \left\| \langle y \rangle^{-\sigma/2} \partial_x \Lambda^{-2/3} f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + C_{\varepsilon_2} \left\| \langle y \rangle^{-\sigma/2} f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \\
\leq{}& \varepsilon_2 \left\| (\partial_y u)^{1/2}\, \partial_x \Lambda^{-2/3} f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + C_{\varepsilon_2} \left\| f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}.
\end{aligned}
$$

From the above inequalities, we infer that, choosing ε_2 small enough,

$$
\begin{aligned}
&\left\| (\partial_y u)^{1/2}\, \partial_x \Lambda^{-2/3} f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\| \partial_y^2 \Lambda^{-1/3} f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \\
\leq{}& \varepsilon_1 \int_0^T \int_{\mathbb{R}} \left| \left(\partial_y^2 \Lambda^{-1/2} f \right)(t,x,0) \right|^2 dx dt \\
&+ C_{\varepsilon_1} \left(\left\| \langle y \rangle^{-\sigma/2} \partial_y \Lambda^{1/6} f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\| \langle y \rangle^{\sigma} \Lambda^{-1/3} \partial_y g \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \right) \\
&+ C_{\varepsilon_1} \left(\left\| \partial_y f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\| f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\| \Lambda^{-1/3} h \right\|^2_{L^2(\mathbb{R}^3_+)} \right).
\end{aligned}
\tag{3.15}
$$

Step (5). In this step we treat the first term on the right side of (3.15), and show that, for any $0 < \varepsilon < 1$,

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}} \left| \left(\partial_y^2 \Lambda^{-1/2} f \right)(t,x,0) \right|^2 dx dt \\
\leq{}& C \left\| (\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f \right\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + \varepsilon C \left\| \Lambda^{-2/3} \partial_y h \right\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&+ C_{\varepsilon} \left(\left\| f \right\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + \left\| \partial_y f \right\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \right).
\end{aligned}
\tag{3.16}
$$

To do so, we integrate by parts to get

$$
\begin{aligned}
\int_0^T \int_{\mathbb{R}} \left| \left(\partial_y^2 \Lambda^{-1/2} f \right)(t,x,0) \right|^2 dx dt &= 2\mathrm{Re}\ \left(\partial_y^3 \Lambda^{-1/2} f,\ \partial_y^2 \Lambda^{-1/2} f \right)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&= 2\mathrm{Re}\ \left(\partial_y^3 \Lambda^{-2/3} f,\ \partial_y^2 \Lambda^{-1/3} f \right)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}.
\end{aligned}
$$

This yields

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}}\left|\left(\partial_y^2\Lambda^{-1/2}f\right)(t,x,0)\right|^2dxdt\\ &\leq\frac{\varepsilon}{2}\left\|\partial_y^3\Lambda^{-2/3}f\right\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}+2\varepsilon^{-1}\left\|\partial_y^2\Lambda^{-1/3}f\right\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\ &\leq\varepsilon\left\|\partial_y^3\Lambda^{-2/3}f\right\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}+C_\varepsilon\left\|\partial_y f\right\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)},\end{aligned}\tag{3.17}$$

the last inequality holding because we can use (2.21) to integrate by parts and then obtain

$$\begin{aligned}&\left\|\partial_y^2\Lambda^{-1/3}f\right\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}=\left(\partial_y^2\Lambda^{-2/3}f,\ \partial_y^2 f\right)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\ &\leq\left|\left(\partial_y^3\Lambda^{-2/3}f,\ \partial_y f\right)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\right|\\ &\leq\left\|\partial_y^3\Lambda^{-2/3}f\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\left\|\partial_y f\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}.\end{aligned}\tag{3.18}$$

Thus in order to prove (3.16) it suffices to estimate $\left\|\partial_y^3\Lambda^{-2/3}f\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}$. We study the equation

$$\begin{aligned}&\partial_t\Lambda^{-2/3}\partial_y f+u\partial_x\Lambda^{-2/3}\partial_y f+v\partial_y\Lambda^{-2/3}\partial_y f-\partial_y^3\Lambda^{-2/3}f\\ =&\ \Lambda^{-2/3}\partial_y h+\left[u\partial_x+v\partial_y,\ \Lambda^{-2/3}\right]\partial_y f-\Lambda^{-2/3}(\partial_y u)\partial_x f-\Lambda^{-2/3}(\partial_y v)\partial_y f,\end{aligned}$$

which implies, by taking the L^2 inner product with $-\partial_y^3\Lambda^{-2/3}f$,

$$\begin{aligned}\left\|\partial_y^3\Lambda^{-2/3}f\right\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}=&-\operatorname{Re}\ \left(\partial_t\Lambda^{-2/3}\partial_y f,\ \partial_y^3\Lambda^{-2/3}f\right)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\ &-\operatorname{Re}\ \left(u\partial_x\Lambda^{-2/3}\partial_y f+v\partial_y\Lambda^{-2/3}\partial_y f,\ \partial_y^3\Lambda^{-2/3}f\right)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\ &+\operatorname{Re}\ \left(\Lambda^{-2/3}\partial_y h,\ \partial_y^3\Lambda^{-2/3}f\right)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\ &+\operatorname{Re}\ \left(\left[u\partial_x+v\partial_y,\ \Lambda^{-2/3}\right]\partial_y f,\ \partial_y^3\Lambda^{-2/3}f\right)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\ &-\operatorname{Re}\ \left(\Lambda^{-2/3}(\partial_y u)\partial_x f,\ \partial_y^3\Lambda^{-2/3}f\right)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\ &-\operatorname{Re}\ \left(\Lambda^{-2/3}(\partial_y v)\partial_y f,\ \partial_y^3\Lambda^{-2/3}f\right)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}.\end{aligned}\tag{3.19}$$

Next we will treat the terms on the right-hand side. Observing

$$\partial_t\Lambda^{-2/3}\partial_y f\big|_{y=0}=0$$

due to (2.21), we integrate by parts to compute

$$\begin{aligned}&-\operatorname{Re}\ \left(\partial_t\Lambda^{-2/3}\partial_y f,\ \ -\partial_y^3\Lambda^{-2/3}f\right)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\ =&-\operatorname{Re}\ \left(\partial_t\partial_y^2\Lambda^{-2/3}f,\ \partial_y^2\Lambda^{-2/3}f\right)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\ =&\ 0,\end{aligned}$$

the last equality holding because

$$\partial_y^2 \Lambda^{-2/3} f\big|_{t=0} = \partial_y^2 \Lambda^{-2/3} f\big|_{t=T} = 0$$

due to (2.20). Since $u\big|_{y=0}$ then integrating by parts gives

$$\begin{aligned}
& - \text{ Re } \Big(u\partial_x \Lambda^{-2/3}\partial_y f,\ -\partial_y^3 \Lambda^{-2/3} f\Big)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&= - \text{ Re } \Big(u\partial_x \Lambda^{-2/3}\partial_y^2 f,\ \partial_y^2 \Lambda^{-2/3} f\Big)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&\quad - \text{ Re } \Big((\partial_y u)\partial_x \Lambda^{-2/3}\partial_y f,\ \partial_y^2 \Lambda^{-2/3} f\Big)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&= \frac{1}{2}\Big((\partial_x u)\Lambda^{-2/3}\partial_y^2 f,\ \Lambda^{-2/3}\partial_y^2 f\Big)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&\quad - \text{ Re } \Big((\partial_y u)\partial_x \Lambda^{-2/3}\partial_y f,\ \partial_y^2 \Lambda^{-2/3} f\Big)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&\le \frac{1}{2}\big\|\partial_x u\big\|_{L^\infty}\big\|\Lambda^{-2/3}\partial_y^2 f\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&\quad + \big\|\Lambda^{-1/3}(\partial_y u)\partial_x \Lambda^{-2/3}\partial_y f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + \big\|\partial_y^2 \Lambda^{-1/3} f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}.
\end{aligned}$$

On the other hand, using Lemma 3.1 gives

$$\begin{aligned}
& \big\|\Lambda^{-1/3}(\partial_y u)\partial_x \Lambda^{-2/3}\partial_y f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&\le 2\big\|\Lambda^{-1/3}\partial_x \Lambda^{-2/3}(\partial_y u)\partial_y f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + 2\big\|\Lambda^{-1/3}\big[\partial_y u,\ \partial_x \Lambda^{-2/3}\big]\partial_y f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&\le C\big\|\partial_y f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}.
\end{aligned}$$

Thus

$$\begin{aligned}
& - \text{ Re } \Big(u\partial_x \Lambda^{-2/3}\partial_y f,\ -\partial_y^3 \Lambda^{-2/3} f\Big)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&\le C\ \Big(\big\|\partial_y f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + \big\|\partial_y^2 \Lambda^{-1/3} f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\Big) \\
&\le \tilde\varepsilon\ \big\|\partial_y^3 \Lambda^{-2/3} f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + C_{\tilde\varepsilon}\big\|\partial_y f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)},
\end{aligned}$$

where the last inequality uses (3.18). Using (3.18) we conclude

$$\begin{aligned}
& - \text{ Re } \Big(v\partial_y \Lambda^{-2/3}\partial_y f,\ -\partial_y^3 \Lambda^{-2/3} f\Big)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&\le\ \frac{\tilde\varepsilon}{2}\big\|\partial_y^3 \Lambda^{-2/3} f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + C_{\tilde\varepsilon}\big\|\partial_y^2 \Lambda^{-1/3} f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&\le\ \tilde\varepsilon\ \big\|\partial_y^3 \Lambda^{-2/3} f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + C_{\tilde\varepsilon}\big\|\partial_y f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}.
\end{aligned}$$

The Cauchy–Schwarz inequality gives, for any $\tilde\varepsilon > 0$,

$$\begin{aligned}
& \text{Re } \Big(\Lambda^{-2/3}\partial_y h,\ -\partial_y^3 \Lambda^{-2/3} f\Big)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&\le \tilde\varepsilon\big\|\partial_y^3 \Lambda^{-2/3} f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + \tilde\varepsilon^{-1}\big\|\Lambda^{-2/3}\partial_y h\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}, \\
& - \text{ Re } \Big(\Lambda^{-2/3}(\partial_y v)\partial_y f,\ -\partial_y^3 \Lambda^{-2/3} f\Big)_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&\le \tilde\varepsilon\big\|\partial_y^3 \Lambda^{-2/3} f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} + \tilde\varepsilon^{-1}\big\|\partial_y v\big\|^2_{L^\infty}\big\|\partial_y f\big\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)},
\end{aligned}$$

and

$$
\begin{aligned}
&\mathrm{Re}\ \Big(\big[u\partial_x+v\partial_y,\ \Lambda^{-2/3}\big]\partial_y f,\ \ -\partial_y^3\Lambda^{-2/3}f\Big)_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
\le&\frac{\tilde\varepsilon}{2}\big\|\partial_y^3\Lambda^{-2/3}f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
&+\ 2\tilde\varepsilon^{-1}\big\|\big[u\partial_x+v\partial_y,\ \Lambda^{-2/3}\big]\partial_y f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
\le&\ \frac{\tilde\varepsilon}{2}\big\|\partial_y^3\Lambda^{-2/3}f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}+C_{\tilde\varepsilon}\big\|\partial_y f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}+C_{\tilde\varepsilon}\big\|\partial_y^2\Lambda^{-1/3}f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
\le&\ \tilde\varepsilon\ \big\|\partial_y^3\Lambda^{-2/3}f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}+C_{\tilde\varepsilon}\big\|\partial_y f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)},
\end{aligned}\tag{3.20}
$$

the second inequality using Lemma 3.1, with the last inequality following from (3.18). Finally,

$$
\begin{aligned}
&-\ \mathrm{Re}\ \Big(\Lambda^{-2/3}(\partial_y u)\partial_x f,\ \ -\partial_y^3\Lambda^{-2/3}f\Big)_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
\le&\ \tilde\varepsilon\big\|\partial_y^3\Lambda^{-2/3}f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}+\tilde\varepsilon^{-1}\big\|\Lambda^{-2/3}(\partial_y u)\partial_x f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
\le&\ \tilde\varepsilon\big\|\partial_y^3\Lambda^{-2/3}f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}+\tilde\varepsilon^{-1}\big\|(\partial_y u)\partial_x\Lambda^{-2/3}f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
&+\tilde\varepsilon^{-1}\big\|\big[\partial_y u,\ \Lambda^{-2/3}\big]\partial_x f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
\le&\ \tilde\varepsilon\big\|\partial_y^3\Lambda^{-2/3}f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}+C_{\tilde\varepsilon}\big\|(\partial_y u)^{1/2}\partial_x\Lambda^{-2/3}f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
&+C_{\tilde\varepsilon}\big\|f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}.
\end{aligned}
$$

This, along with (3.19)–(3.20), yields, for any $\tilde\varepsilon>0$,

$$
\begin{aligned}
&\big\|\partial_y^3\Lambda^{-2/3}f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
\le&\ \tilde\varepsilon\big\|\partial_y^3\Lambda^{-2/3}f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}+C_{\tilde\varepsilon}\big\|(\partial_y u)^{1/2}\partial_x\Lambda^{-2/3}f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
&+C_{\tilde\varepsilon}\left(\big\|\Lambda^{-2/3}\partial_y h\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}+\big\|\partial_y f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}+\big\|f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\right).
\end{aligned}
$$

Thus letting $\tilde\varepsilon$ be small enough, we have

$$
\begin{aligned}
&\big\|\partial_y^3\Lambda^{-2/3}f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
&\le C\big\|(\partial_y u)^{1/2}\partial_x\Lambda^{-2/3}f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
&\quad+\ C\left(\big\|\Lambda^{-2/3}\partial_y h\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}+\big\|f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}+\big\|\partial_y f\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\right).
\end{aligned}\tag{3.21}
$$

This along with (3.17) yields the desired estimate (3.16).

Step (6). Now we combine (3.15) and (3.16) to conclude, for any $0<\varepsilon,\varepsilon_1<1$,

$$
\begin{aligned}
&\big\|\,(\partial_y u)^{1/2}\,\partial_x\Lambda^{-2/3}f\big\|^2_{L^2([0,T]\times\mathbb{R}_+^2)}+\big\|\partial_y^2\Lambda^{-1/3}f\big\|^2_{L^2([0,T]\times\mathbb{R}_+^2)}\\
\le&\ \varepsilon_1 C\big\|\,(\partial_y u)^{1/2}\,\partial_x\Lambda^{-2/3}f\big\|^2_{L^2([0,T]\times\mathbb{R}_+^2)}+\varepsilon_1\varepsilon C\big\|\Lambda^{-2/3}\partial_y h\big\|^2_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
&+C_{\varepsilon_1,\varepsilon}\left(\big\|\,\langle y\rangle^{-\sigma/2}\,\partial_y\Lambda^{1/6}f\big\|^2_{L^2([0,T]\times\mathbb{R}_+^2)}+\big\|\,\langle y\rangle^{\sigma}\,\Lambda^{-1/3}\partial_y g\big\|^2_{L^2([0,T]\times\mathbb{R}_+^2)}\right)\\
&+C_{\varepsilon_1,\varepsilon}\left(\big\|\partial_y f\big\|^2_{L^2([0,T]\times\mathbb{R}_+^2)}+\big\|f\big\|^2_{L^2([0,T]\times\mathbb{R}_+^2)}+\big\|\Lambda^{-1/3}h\big\|^2_{L^2(\mathbb{R}_+^3)}\right),
\end{aligned}
$$

which implies, choosing $\varepsilon_1 > 0$ sufficiently small,

$$\begin{aligned}
&\left\| (\partial_y u)^{1/2} \partial_x \Lambda^{-2/3} f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\| \partial_y^2 \Lambda^{-1/3} f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \\
\le\ & \varepsilon \left\| \Lambda^{-2/3} \partial_y h \right\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&+ C_\varepsilon \left(\left\| \langle y \rangle^{-\sigma/2} \partial_y \Lambda^{1/6} f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\| \langle y \rangle^{\sigma} \Lambda^{-1/3} \partial_y g \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \right) \\
&+ C_\varepsilon \left(\left\| \partial_y f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\| f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\| \Lambda^{-1/3} h \right\|^2_{L^2(\mathbb{R}^3_+)} \right)
\end{aligned}$$

with $\varepsilon > 0$ arbitrarily small. This, along with

$$\begin{aligned}
\left\| \langle y \rangle^{-\sigma/2} \Lambda^{1/3} f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} &\le C \left\| (\partial_y u)^{1/2} \Lambda^{1/3} f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \\
&\le C \left\| \langle y \rangle^{-\sigma/2} \partial_x \Lambda^{-2/3} f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + C \left\| f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)}
\end{aligned}$$

due to (1.2), implies, for any $\varepsilon > 0$,

$$\begin{aligned}
&\left\| \langle y \rangle^{-\sigma/2} \Lambda^{1/3} f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\| \partial_y^2 \Lambda^{-1/3} f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \\
\le\ & \varepsilon \left\| \Lambda^{-2/3} \partial_y h \right\|^2_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \\
&+ C_\varepsilon \left(\left\| \langle y \rangle^{-\sigma/2} \partial_y \Lambda^{1/6} f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\| \langle y \rangle^{\sigma} \Lambda^{-1/3} \partial_y g \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} \right) \\
&+ C_\varepsilon \left(\left\| \partial_y f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\| f \right\|^2_{L^2([0,T]\times\mathbb{R}^2_+)} + \left\| \Lambda^{-1/3} h \right\|^2_{L^2(\mathbb{R}^3_+)} \right).
\end{aligned}$$

This is just the first estimate in Proposition 2.4. And the second estimate follows from (3.21) since $|\partial_y u|$ is bounded from above by $\langle y \rangle^{-\sigma}$. Thus the proof of Proposition 2.4 is complete. ∎

4. Property of inductive weight functions. This section is devoted to proving Lemmas 2.1, 2.2, and 2.5, used in section 2.

Recall, for $m \ge N_0 + 1$ and $0 \le \ell \le 3, y > 0, 0 \le t \le T < 1$,

$$W_m^\ell = e^{2cy} \left(1 + \frac{2cy}{(3m+\ell)\sigma} \right)^{-\frac{(3m+\ell)\sigma}{2}} (1+cy)^{-1} \Lambda^{\frac{\ell}{3}}, \qquad \phi_m^\ell = \phi^{3(m-N_0-1)+\ell},$$

thus

$$\phi_{m_1}^{\ell_1} \le \phi_{m_2}^{\ell_2} \tag{4.1}$$

provided $N_0 + 1 \le m_2 \le m_1$ and $0 \le \ell_2 \le \ell_1 \le 3$.

Next we list some inequalities for the weight W_m^ℓ. Observe the function

$$\gamma \longrightarrow \left(1 + \frac{cy}{\gamma} \right)^{-\gamma}$$

is a monotonically decreasing function as γ varies in the interval $[1, +\infty[$ for $y \ge 0$. Thus

$$0 \le \ell \le 3, \quad \left\| W_{m_1}^\ell f \right\|_{L^2(\mathbb{R}_x)} \le \left\| W_{m_2}^\ell f \right\|_{L^2(\mathbb{R}_x)}, \tag{4.2}$$

and

$$\forall\ 0 \le \ell \le i \le 3, \quad \left\|W_{m_1}^i f\right\|_{L^2(\mathbb{R}_x)} \le \left\|W_{m_2}^{i-\ell}\Lambda^{\ell/3} f\right\|_{L^2(\mathbb{R}_x)} \le \left\|W_{m_3}^i f\right\|_{L^2(\mathbb{R}_x)}, \tag{4.3}$$

provided that $m_1 \ge m_2 \ge 1$, and that $3m_2 + i - \ell \ge 3m_3 + i$. Moreover, since

$$\forall\ 0 \le \alpha \le 3,\ \forall\ \gamma \ge 1$$
$$\left|\partial_y^\alpha e^{2cy}\left(1+\frac{cy}{\gamma}\right)^{-\gamma}(1+cy)^{-1}\right| \le C_\alpha e^{2cy}\left(1+\frac{cy}{\gamma}\right)^{-\gamma}(1+cy)^{-1}$$

with C_α a constant independent of γ, then the following estimates,

$$\left\|[\partial_y,\ W_m^i]f\right\|_{L^2(\mathbb{R}_+^2)} \le C\left\|W_m^i f\right\|_{L^2(\mathbb{R}_+^2)}, \tag{4.4}$$

$$\begin{aligned}\left\|[\partial_y^2,\ W_m^i]f\right\|_{L^2(\mathbb{R}_+^2)} &\le C\left(\left\|W_m^i f\right\|_{L^2(\mathbb{R}_+^2)} + \left\|W_m^i \partial_y f\right\|_{L^2(\mathbb{R}_+^2)}\right)\\ &\le \tilde{C}\left(\left\|W_m^i f\right\|_{L^2(\mathbb{R}_+^2)} + \left\|\partial_y W_m^i f\right\|_{L^2(\mathbb{R}_+^2)}\right),\end{aligned} \tag{4.5}$$

$$\begin{aligned}\left\|[\partial_y^3,\ W_m^i]f\right\|_{L^2(\mathbb{R}_+^2)} &\le C\left(\left\|W_m^i f\right\|_{L^2(\mathbb{R}_+^2)} + \left\|W_m^i \partial_y f\right\|_{L^2(\mathbb{R}_+^2)} + \left\|W_m^i \partial_y^2 f\right\|_{L^2(\mathbb{R}_+^2)}\right)\\ &\le \tilde{C}\left(\left\|W_m^i f\right\|_{L^2(\mathbb{R}_+^2)} + \left\|\partial_y W_m^i f\right\|_{L^2(\mathbb{R}_+^2)} + \left\|\partial_y^2 W_m^i f\right\|_{L^2(\mathbb{R}_+^2)}\right)\end{aligned} \tag{4.6}$$

hold for all integers m, i with $m \ge 1$ and $0 \le i \le 3$, where $C, \tilde{C}$ are two constants independent of m.

Lemma 4.1. *Under the assumption* (1.2) *and* (1.3)*, let c be the constant given in* (2.2)*, and $\Lambda^{\tau_1}, \Lambda_\delta^{\tau_2}$ be the Fourier multiplier associate with the symbols $\langle\xi\rangle^{\tau_1}$ and $\langle\delta\xi\rangle^{\tau_2}$, respectively. Then there exists a constant C, such that for any $m, n \ge 1, 0 \le \ell \le 3$, and for any $0 < \tilde{c} < c$, we have*

$$\left\|e^{\tilde{c}y}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\partial_x^m u\right\|_{L^2(\mathbb{R}_+^2)} \le C\left\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}W_n^\ell f_m\right\|_{L^2(\mathbb{R}_+^2)} \tag{4.7}$$

and

$$\left\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\partial_x^m v\right\|_{L^\infty(\mathbb{R}_+;\ L^2(\mathbb{R}_x))} \le C\left\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}W_n^\ell f_{m+1}\right\|_{L^2(\mathbb{R}_+^2)}. \tag{4.8}$$

Proof. In the proof we use C to denote different constants which are independent of m. Observe $\omega \in L^\infty$ and $\omega > 0$, then

$$\left\|e^{\tilde{c}y}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\partial_x^m u\right\|_{L^2(\mathbb{R}_+^2)} \le C\left\|e^{\tilde{c}y}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\frac{\partial_x^m u}{\omega}\right\|_{L^2(\mathbb{R}_+^2)}.$$

On the other hand, integrating by parts we have

$$\begin{aligned}\left\|e^{\tilde{c}y}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\frac{\partial_x^m u}{\omega}\right\|_{L^2(\mathbb{R}_+^2)}^2 &= \int_{\mathbb{R}}\int_0^\infty e^{2\tilde{c}y}\left(\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\frac{\partial_x^m u}{\omega}\right)\overline{\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\frac{\partial_x^m u}{\omega}}dydx\\ &= \frac{1}{2\tilde{c}}\int_{\mathbb{R}}\int_0^\infty\left(\partial_y e^{2\tilde{c}y}\right)\left(\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\frac{\partial_x^m u}{\omega}\right)\overline{\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\frac{\partial_x^m u}{\omega}}dydx\\ &= -\frac{1}{2\tilde{c}}\int_{\mathbb{R}}\int_0^\infty e^{2\tilde{c}y}\left[\partial_y\left(\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\frac{\partial_x^m u}{\omega}\right)\right]\overline{\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\frac{\partial_x^m u}{\omega}}dydx\\ &\quad -\frac{1}{2\tilde{c}}\int_{\mathbb{R}}\int_0^\infty e^{2\tilde{c}y}\left(\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\frac{\partial_x^m u}{\omega}\right)\overline{\partial_y\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\frac{\partial_x^m u}{\omega}}dydx\\ &\le \frac{1}{\tilde{c}}\left\|e^{\tilde{c}y}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\frac{\partial_x^m u}{\omega}\right\|_{L^2(\mathbb{R}_+^2)}\left\|e^{\tilde{c}y}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\right\|_{L^2(\mathbb{R}_+^2)},\end{aligned}$$

which implies

$$\begin{aligned}\left\|e^{\tilde{c}y}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\frac{\partial_x^m u}{\omega}\right\|_{L^2(\mathbb{R}_+^2)} &\leq \left\|e^{\tilde{c}y}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\right\|_{L^2(\mathbb{R}_+^2)}\\ &= \left\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}e^{\tilde{c}y}\omega^{-1}\omega\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\right\|_{L^2(\mathbb{R}_+^2)}\\ &\leq \left\|e^{\tilde{c}y}\omega^{-1}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\omega\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\right\|_{L^2(\mathbb{R}_+^2)}\\ &\quad+\left\|\left[e^{\tilde{c}y}\omega^{-1},\ \Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\right]\omega\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\right\|_{L^2(\mathbb{R}_+^2)}.\end{aligned}$$

Thus we have, by the above inequalities,

$$\begin{aligned}\left\|e^{\tilde{c}y}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\partial_x^m u\right\|_{L^2(\mathbb{R}_+^2)} &\leq C\left\|e^{\tilde{c}y}\omega^{-1}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\omega\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\right\|_{L^2(\mathbb{R}_+^2)}\\ &\quad+C\left\|\left[e^{\tilde{c}y}\omega^{-1},\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\right]\omega\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\right\|_{L^2(\mathbb{R}_+^2)}.\end{aligned}$$

On the other hand, (1.2) and (1.3) enable us to use Lemma 3.1 to obtain

$$\begin{aligned}\left\|\left[e^{\tilde{c}y}\omega^{-1},\ \Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\right]\omega\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\right\|_{L^2(\mathbb{R}_+^2)} &\leq C\left\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\omega\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\right\|_{L^2(\mathbb{R}_+^2)}\\ &\leq C\left\|e^{\tilde{c}y}\omega^{-1}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\omega\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\right\|_{L^2(\mathbb{R}_+^2)}.\end{aligned}$$

As a result,

$$\begin{aligned}&\left\|e^{\tilde{c}y}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\partial_x^m u\right\|_{L^2(\mathbb{R}_+^2)} \leq C\left\|e^{\tilde{c}y}\omega^{-1}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\omega\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\right\|_{L^2(\mathbb{R}_+^2)}\\ &\leq C\left\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}W_n^\ell f_m\right\|_{L^2(\mathbb{R}_+^2)},\end{aligned}$$

the last inequality using the fact that $f_m = \omega\partial_y\left(\frac{\partial_x^m u}{\omega}\right)$ and that

$$e^{\tilde{c}y}\omega^{-1} \leq Ce^{\tilde{c}y}(1+y)^\sigma \leq Ce^{2cy}\left(1+\frac{2cy}{\gamma}\right)^{-\gamma/2}$$

for any $\gamma \geq 1$. This is just the desired (4.7). Now we prove (4.8). Recall $v(t,x,y) = -\int_0^y \partial_x u(t,x,y')dy'$. Then we have

$$\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\partial_x^m v = -\int_0^y \Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\partial_x^{m+1}u(x,y')dy'.$$

Therefore

$$\begin{aligned}\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\partial_x^m v\|_{L^\infty(\mathbb{R}_+;\ L^2(\mathbb{R}_x))} &\leq \left\|e^{-\tilde{c}y}\right\|_{L^2(\mathbb{R}_+)}\left\|e^{\tilde{c}y}\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}\partial_x^{m+1}u\right\|_{L^2\left(\mathbb{R}_+^2\right)}\\ &\leq C\left\|\Lambda^{\tau_1}\Lambda_\delta^{\tau_2}W_n^\ell f_{m+1}\right\|_{L^2\left(\mathbb{R}_+^2\right)},\end{aligned}$$

the last inequality using (4.7). Thus the desired (4.8) follows and the proof of Lemma 4.1 is complete. ∎

We prove now Lemma 2.1; recall

$$f_m = \partial_x^m \omega - \frac{\partial_y \omega}{\omega} \partial_x^m u = \omega \partial_y \left(\frac{\partial_x^m u}{\omega} \right).$$

LEMMA 4.2. *There exists a constant C such that*

$$\| \langle y \rangle^{-1} W_m^\ell \partial_x^m u \|_{L^2(\mathbb{R}_+^2)} + \| \langle y \rangle^{-1} W_m^\ell \partial_x^m \omega \|_{L^2(\mathbb{R}_+^2)} \leq C \| W_m^\ell f_m \|_{L^2(\mathbb{R}_+^2)}. \tag{4.9}$$

As a result, for some constant $\tilde{C}$,

$$\| \Lambda^{-1} W_m^0 f_{m+1} \|_{L^2(\mathbb{R}_+^2)} \leq \tilde{C} \| W_m^0 f_m \|_{L^2(\mathbb{R}_+^2)}$$

and

$$\| \Lambda^{-1} \partial_y W_m^0 f_{m+1} \|_{L^2(\mathbb{R}_+^2)} \leq \tilde{C} \left(\| \partial_y W_m^0 f_m \|_{L^2(\mathbb{R}_+^2)} + \| W_m^0 f_m \|_{L^2(\mathbb{R}_+^2)} \right).$$

Proof. In the proof we use C to denote different constants which depend only on σ, c, and C_* and are independent of m. We first prove (4.9). Observe

$$\begin{aligned}
&\omega \langle y \rangle^{-1} \left(1 + \frac{2cy}{(3m+\ell)\sigma} \right)^{-(3m+\ell)\sigma/2} (1+cy)^{-1} \\
&\leq C(1+y)^{-\sigma-1} \left(1 + \frac{2cy}{(3m+\ell)\sigma} \right)^{-(3m+\ell)\sigma/2} \\
&\leq \; C R^{\sigma+1} (R+y)^{-\sigma-1} \left(1 + \frac{2cy}{(3m+\ell)\sigma} \right)^{-(3m+\ell)\sigma/2},
\end{aligned}$$

where $R \geq 1$ is a large number to be determined later. Thus using the notation

$$b_{m,\ell}^R(y) = \left(1 + \frac{2cy}{(3m+\ell)\sigma} \right)^{-(3m+\ell)\sigma/2} (R+y)^{-\sigma-1},$$

we have

$$\begin{aligned}
\| \langle y \rangle^{-1} W_m^\ell \partial_x^m u \|_{L^2(\mathbb{R}_+^2)} &= \| \langle y \rangle^{-1} W_m^\ell (\omega \frac{\partial_x^m u}{\omega}) \|_{L^2(\mathbb{R}_+^2)} \\
&\leq \| \omega \langle y \rangle^{-1} W_m^\ell \frac{\partial_x^m u}{\omega} \|_{L^2(\mathbb{R}_+^2)} + \| \langle y \rangle^{-1} \left[W_m^\ell, \; \omega \right] \frac{\partial_x^m u}{\omega} \|_{L^2(\mathbb{R}_+^2)} \\
&\leq C R^{\sigma+1} \| e^{2cy} b_{m,\ell}^R \frac{\Lambda^{\ell/3} \partial_x^m u}{\omega} \|_{L^2(\mathbb{R}_+^2)} \\
&\quad + \| \langle y \rangle^{-1} \left[W_m^\ell, \; \omega \right] \frac{\partial_x^m u}{\omega} \|_{L^2(\mathbb{R}_+^2)}.
\end{aligned}$$

On the other hand, using Lemma 3.1

$$\begin{aligned}
&\| \langle y \rangle^{-1} \left[W_m^\ell, \; \omega \right] \frac{\partial_x^m u}{\omega} \|_{L^2(\mathbb{R}_+^2)} \\
&\leq R \| \left[\Lambda^{\frac{l}{3}}, \; \omega \right] e^{2cy} b_{m,\ell}^R \frac{\partial_x^m u}{\omega} \|_{L^2(\mathbb{R}_+^2)} \leq C R \| e^{2cy} b_{m,\ell}^R \frac{\partial_x^m u}{\omega} \|_{L^2(\mathbb{R}_+^2)}.
\end{aligned}$$

Combining these inequalities we conclude

$$\left\| \langle y \rangle^{-1} W_m^\ell \partial_x^m u \right\|_{L^2(\mathbb{R}_+^2)} \leq C R^{\sigma+1} \left\| e^{2cy} b_{m,\ell}^R \Lambda^{\ell/3} \frac{\partial_x^m u}{\omega} \right\|_{L^2(\mathbb{R}_+^2)}. \tag{4.10}$$

Moreover, observe $u\big|_{y=0} = 0$ and thus we have, integrating parts,

$$\begin{aligned}
&\big\| e^{2cy} b_{m,\ell}^R \Lambda^{\ell/3} \frac{\partial_x^m u}{\omega} \big\|_{L^2(\mathbb{R}_+^2)}^2 \\
&= \int_{\mathbb{R}} \int_0^\infty e^{4cy} \left(b_{m,\ell}^R(y) \right)^2 \left(\Lambda^{\ell/3} \frac{\partial_x^m u}{\omega} \right) \overline{\Lambda^{\ell/3} \frac{\partial_x^m u}{\omega}} dydx \\
&= \frac{1}{4c} \int_{\mathbb{R}} \int_0^\infty \left(\partial_y e^{4cy} \right) \left(b_{m,\ell}^R(y) \right)^2 \left(\Lambda^{\ell/3} \frac{\partial_x^m u}{\omega} \right) \overline{\Lambda^{\ell/3} \frac{\partial_x^m u}{\omega}} dydx \\
&= -\frac{1}{2c} \int_{\mathbb{R}} \int_0^\infty e^{4cy} b_{m,\ell}^R(y) \left(\partial_y b_{m,\ell}^R(y) \right) \left(\Lambda^{\ell/3} \frac{\partial_x^m u}{\omega} \right) \overline{\Lambda^{\ell/3} \frac{\partial_x^m u}{\omega}} dydx \\
&\quad - \frac{1}{4c} \int_{\mathbb{R}} \int_0^\infty e^{4cy} \left(b_{m,\ell}^R(y) \right)^2 \left[\partial_y \left(\Lambda^{\ell/3} \frac{\partial_x^m u}{\omega} \right) \right] \overline{\Lambda^{\ell/3} \frac{\partial_x^m u}{\omega}} dydx \\
&\quad - \frac{1}{4c} \int_{\mathbb{R}} \int_0^\infty e^{4cy} \left(b_{m,\ell}^R(y) \right)^2 \left(\Lambda^{\ell/3} \frac{\partial_x^m u}{\omega} \right) \overline{\partial_y \Lambda^{\ell/3} \left(\frac{\partial_x^m u}{\omega} \right)} dydx,
\end{aligned}$$

which, along with the estimate

$$\left| \partial_y b_{m,\ell}^R \right| \leq \left(c + (\sigma+1) R^{-1} \right) b_{m,\ell}^R,$$

gives

$$\begin{aligned}
\big\| e^{2cy} b_{m,\ell}^R \Lambda^{\ell/3} \frac{\partial_x^m u}{\omega} \big\|_{L^2(\mathbb{R}_+^2)}^2 &\leq \frac{c + (\sigma+1) R^{-1}}{2c} \left\| e^{2cy} b_{m,\ell}^R \Lambda^{\ell/3} \frac{\partial_x^m u}{\omega} \right\|_{L^2(\mathbb{R}_+^2)}^2 \\
&\quad + \frac{1}{2c} \left\| e^{2cy} b_{m,\ell}^R \Lambda^{\ell/3} \frac{\partial_x^m u}{\omega} \right\|_{L^2(\mathbb{R}_+^2)} \left\| e^{2cy} b_{m,\ell}^R \partial_y \left(\Lambda^{\ell/3} \frac{\partial_x^m u}{\omega} \right) \right\|_{L^2(\mathbb{R}_+^2)}.
\end{aligned}$$

Now we choose $R = 1 + 2(\sigma+1)c^{-1}$, which gives $R \geq 1$ and

$$(\sigma+1) R^{-1} \leq \frac{c}{2}.$$

Then we deduce, from the above inequalities,

$$\left\| e^{2cy} b_{m,\ell}^R \Lambda^{\ell/3} \frac{\partial_x^m u}{\omega} \right\|_{L^2(\mathbb{R}_+^2)} \leq \frac{2}{c} \left\| e^{2cy} b_{m,\ell}^R \partial_y \Lambda^{\ell/3} \left(\frac{\partial_x^m u}{\omega} \right) \right\|_{L^2(\mathbb{R}_+^2)}.$$

Moreover, observe $R \geq c^{-1} + 1$ and the monotonicity assumption $\omega \geq C_*^{-1}(1+y)^{-\sigma}$, and thus

$$\begin{aligned}
b_{m,\ell}^R &\leq c(1+y)^{-\sigma} (1+cy)^{-1} \left(1 + \frac{2cy}{(3m+\ell)\sigma} \right)^{-(3m+\ell)\sigma/2} \\
&\leq c\, C_* \omega (1+cy)^{-1} \left(1 + \frac{2cy}{(3m+\ell)\sigma} \right)^{-(3m+\ell)\sigma/2}.
\end{aligned}$$

As a result, we obtain

$$\left\| e^{2cy} b_{m,\ell}^R \Lambda^{\ell/3} \frac{\partial_x^m u}{\omega} \right\|_{L^2(\mathbb{R}_+^2)} \leq C_* \left\| \omega W_m^\ell \partial_y \left(\frac{\partial_x^m u}{\omega} \right) \right\|_{L^2(\mathbb{R}_+^2)},$$

which along with (4.10) gives

$$
\begin{aligned}
&\left\|\langle y\rangle^{-1} W_m^{\ell}\partial_x^m u\right\|_{L^2(\mathbb{R}_+^2)}\\
&\le C\left\|\omega W_m^{\ell}\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\right\|_{L^2(\mathbb{R}_+^2)}\\
&\le C\left\|W_m^{\ell}\omega\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\right\|_{L^2(\mathbb{R}_+^2)}+C\left\|\left[\omega,\ W_m^{\ell}\right]\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\right\|_{L^2(\mathbb{R}_+^2)}.
\end{aligned}
$$

Using the notation $\rho_{m,\ell}(y)=e^{2cy}\left(1+\frac{2cy}{(3m+\ell)\sigma}\right)^{-(3m+\ell)\sigma/2}(1+cy)^{-1}$,

$$
\begin{aligned}
\left\|\left[\omega,\ W_m^{\ell}\right]\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\right\|_{L^2(\mathbb{R}_+^2)}&=\left\|\left[\omega,\ \Lambda^{\ell/3}\right]\rho_{m,\ell}(y)\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\right\|_{L^2(\mathbb{R}_+^2)}\\
&=\left\|\left[\omega\langle y\rangle^{\sigma},\ \Lambda^{\ell/3}\right]\langle y\rangle^{-\sigma}\rho_{m,\ell}(y)\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\right\|_{L^2(\mathbb{R}_+^2)}\\
&\le C\left\|\langle y\rangle^{-\sigma}\rho_{m,\ell}(y)\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\right\|_{L^2(\mathbb{R}_+^2)}\\
&\le C\left\|\rho_{m,\ell}(y)\omega\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\right\|_{L^2(\mathbb{R}_+^2)}\\
&\le C\left\|W_m^{\ell}\omega\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\right\|_{L^2(\mathbb{R}_+^2)}.
\end{aligned}
$$

Then, combining these inequalities we conclude,

$$
\left\|\langle y\rangle^{-1} W_m^{\ell}\partial_x^m u\right\|_{L^2(\mathbb{R}_+^2)}\le C\left\|W_m^{\ell}\omega\partial_y\left(\frac{\partial_x^m u}{\omega}\right)\right\|_{L^2(\mathbb{R}_+^2)}=C\left\|W_m^{\ell}f_m\right\|_{L^2(\mathbb{R}_+^2)}.
$$

For the other terms in (4.9), we have

$$
\begin{aligned}
&\left\|\langle y\rangle^{-1} W_m^{\ell}\partial_x^m\omega\right\|_{L^2(\mathbb{R}_+^2)}\\
&\le\left\|\langle y\rangle^{-1} W_m^{\ell}f_m\right\|_{L^2(\mathbb{R}_+^2)}+\left\|\langle y\rangle^{-1} W_m^{\ell}\big((\partial_y\omega)/\omega\big)\partial_x^m u\right\|_{L^2(\mathbb{R}_+^2)}\\
&\le\left\|\langle y\rangle^{-1} W_m^{\ell}f_m\right\|_{L^2(\mathbb{R}_+^2)}+\left\|\big((\partial_y\omega)/\omega\big)\langle y\rangle^{-1} W_m^{\ell}\partial_x^m u\right\|_{L^2(\mathbb{R}_+^2)}\\
&\quad+\left\|\left[(\partial_y\omega)/\omega,\ W_m^{\ell}\right]\langle y\rangle^{-1}\partial_x^m u\right\|_{L^2(\mathbb{R}_+^2)}\\
&\le\left\|\langle y\rangle^{-1} W_m^{\ell}f_m\right\|_{L^2(\mathbb{R}_+^2)}+C\left\|\langle y\rangle^{-1} W_m^{\ell}\partial_x^m u\right\|_{L^2(\mathbb{R}_+^2)}\\
&\le C\left\|W_m^{\ell}f_m\right\|_{L^2(\mathbb{R}_+^2)}.
\end{aligned}
$$

Thus the desired estimate (4.9) follows. As a result, we have

$$
\begin{aligned}
\left\|\Lambda^{-1}W_m^0 f_{m+1}\right\|_{L^2(\mathbb{R}_+^2)}&\le\left\|\Lambda^{-1}W_m^0\partial_x f_m\right\|_{L^2(\mathbb{R}_+^2)}+\left\|\Lambda^{-1}W_m^0\left[\partial_x\big((\partial_y\omega)/\omega\big)\right]\partial_x^m u\right\|_{L^2(\mathbb{R}_+^2)}\\
&\le\left\|W_m^0 f_m\right\|_{L^2(\mathbb{R}_+^2)}+\left\|\langle y\rangle^{-1}W_m^0\partial_x^m u\right\|_{L^2(\mathbb{R}_+^2)}\le C\left\|W_m^0 f_m\right\|_{L^2(\mathbb{R}_+^2)}.
\end{aligned}
$$

Similarly, we can deduce that, using (4.4),

$$
\left\|\Lambda^{-1}\partial_y W_m^0 f_{m+1}\right\|_{L^2(\mathbb{R}_+^2)}\le C\left\|\partial_y W_m^0 f_m\right\|_{L^2(\mathbb{R}_+^2)}+\left\|W_m^0 f_m\right\|_{L^2(\mathbb{R}_+^2)}.
$$

Thus the proof of Lemma 4.2 is complete. □

We prove now the Lemma 2.2 by the following 2 lemmas.

LEMMA 4.3. *There exists a constant C such that, for any $m \geq 1$ and $1 \leq \ell \leq 3$,*

$$\| \langle y \rangle^{-\sigma/2} \partial_y \Lambda^{1/3} \Lambda_\delta^{-2} W_m^{\ell-1} f_m \|_{L^2(\mathbb{R}^2_+)} \leq C \| \partial_y \Lambda_\delta^{-2} W_m^\ell f_m \|_{L^2(\mathbb{R}^2_+)} + C \| \Lambda_\delta^{-2} W_m^\ell f_m \|_{L^2(\mathbb{R}^2_+)}.$$

Proof. We can write

$$\Lambda^{1/3} \Lambda_\delta^{-2} W_m^{\ell-1} = e^{2cy} \left(1 + \frac{2cy}{(3m+\ell-1)\sigma}\right)^{-\frac{(3m+\ell-1)\sigma}{2}},$$
$$(1+cy)^{-1} \Lambda^{\frac{\ell}{3}} \Lambda_\delta^{-2} = a_{m,\ell}(y) \Lambda_\delta^{-2} W_m^\ell,$$

where

$$a_{m,\ell}(y) = \left(1 + \frac{2cy}{(3m+\ell-1)\sigma}\right)^{-\frac{(3m+\ell-1)\sigma}{2}} \left(1 + \frac{2cy}{(3m+\ell)\sigma}\right)^{\frac{(3m+\ell)\sigma}{2}}.$$

Direct computation gives

$$\begin{aligned} &|a_{m,\ell}(y)| \\ &= \left(1 + \frac{2cy}{(3m+\ell-1)\sigma}\right)^{\sigma/2} \left(1 + \frac{2cy}{(3m+\ell-1)\sigma}\right)^{-\frac{(3m+\ell)\sigma}{2}} \left(1 + \frac{2cy}{(3m+\ell)\sigma}\right)^{\frac{(3m+\ell)\sigma}{2}} \\ &\leq \left(1 + \frac{2cy}{(3m+\ell-1)\sigma}\right)^{\sigma/2} \leq C \langle y \rangle^{\sigma/2}. \end{aligned}$$

Moreover observe $|\partial_y a_{m,\ell}(y)| \leq 2c\, |a_{m,\ell}(y)|$, and thus

$$|\partial_y a_{m,\ell}(y)| \leq C \langle y \rangle^{\sigma/2}.$$

As a result,

$$\begin{aligned} &\| \langle y \rangle^{-\sigma/2} \partial_y \Lambda^{1/3} \Lambda_\delta^{-2} W_m^{\ell-1} f_m \|_{L^2(\mathbb{R}^2_+)} = \| \langle y \rangle^{-\sigma/2} \partial_y \left(a_{m,\ell} \Lambda_\delta^{-2} W_m^\ell f_m \right) \|_{L^2(\mathbb{R}^2_+)} \\ &\leq \| \langle y \rangle^{-\sigma/2} a_{m,\ell} \partial_y \Lambda_\delta^{-2} W_m^\ell f_m \|_{L^2(\mathbb{R}^2_+)} + \| \langle y \rangle^{-\sigma/2} (\partial_y a_{m,\ell}) \Lambda_\delta^{-2} W_m^\ell f_m \|_{L^2(\mathbb{R}^2_+)} \\ &\leq C \Big(\| \partial_y \Lambda_\delta^{-2} W_m^\ell f_m \|_{L^2(\mathbb{R}^2_+)} + \| \Lambda_\delta^{-2} W_m^\ell f_m \|_{L^2(\mathbb{R}^2_+)} \Big). \end{aligned}$$

The proof of Lemma 4.3 is thus complete. ∎

LEMMA 4.4. *There exists a constant C, depending only on σ, c, and C_*, such that for any integers $m \geq N_0 + 1$, we have*

$$\begin{aligned} &\| \phi^0_{m+1} W^0_{m+1} f_{m+1} \|_{L^\infty([0,T];\ L^2(\mathbb{R}^2_+))} + \sum_{j=1}^{2} \| \partial_y^j \Lambda^{-\frac{2(j-1)}{3}} \phi^0_{m+1} W^0_{m+1} f_{m+1} \|_{L^2([0,T]\times\mathbb{R}^2_+)} \\ &\leq\ C \| \phi^3_m W^3_m f_m \|_{L^\infty([0,T];\ L^2(\mathbb{R}^2_+))} + C \sum_{j=1}^{3} \| \partial_y^j \Lambda^{-\frac{2(j-1)}{3}} \phi^3_m W^3_m f_m \|_{L^2([0,T]\times\mathbb{R}^2_+)} \end{aligned}$$

and

$$
\begin{aligned}
&\left\|\partial_y^3\Lambda^{-1}\phi^0_{m+1}W^0_{m+1}f_{m+1}\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
\leq\ & C\left\|\phi^3_m W^3_m f_m\right\|_{L^\infty([0,T];\ L^2(\mathbb{R}^2_+))} + C\sum_{j=1}^{2}\left\|\partial_y^j\Lambda^{-\frac{2(j-1)}{3}}\phi^3_m W^3_m f_m\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}\\
&+C\left\|\partial_y^3\Lambda^{-1}\phi^3_m W^3_m f_m\right\|_{L^2([0,T]\times\mathbb{R}^2_+)}.
\end{aligned}
$$

Proof. In the proof we use C to denote different constants which are independent of m. In view of the definition (2.1) of f_m, we have, observing (4.1),

$$
\begin{aligned}
&\left\|\phi^0_{m+1}W^0_{m+1}f_{m+1}\right\|_{L^\infty([0,T];\ L^2(\mathbb{R}^2_+))}\\
\leq\ &\left\|\phi^0_{m+1}W^0_{m+1}\partial_x f_m\right\|_{L^\infty([0,T];\ L^2(\mathbb{R}^2_+))}\\
&+\left\|\phi^0_{m+1}W^0_{m+1}\left[\partial_x\big((\partial_y\omega)/\omega\big)\right]\partial_x^m u\right\|_{L^\infty([0,T];\ L^2(\mathbb{R}^2_+))}\\
\leq\ &\left\|\phi^3_m W^0_{m+1}\Lambda^1 f_m\right\|_{L^\infty([0,T];\ L^2(\mathbb{R}^2_+))}\\
&+C\left\|\langle y\rangle^{-1}\phi^3_m W^3_m\partial_x^m u\right\|_{L^\infty([0,T];\ L^2(\mathbb{R}^2_+))}\\
\leq\ & C\left\|\phi^3_m W^3_m f_m\right\|_{L^\infty([0,T];\ L^2(\mathbb{R}^2_+))},
\end{aligned}
$$

the last inequality using (4.9) and (4.3). Similarly, using (4.4), we can deduce that

$$
\begin{aligned}
\left\|\partial_y\phi^0_{m+1}W^0_{m+1}f_{m+1}\right\|_{L^\infty([0,T];\ L^2(\mathbb{R}^2_+))}\leq\ & C\left\|\partial_y\phi^3_m W^3_m f_m\right\|_{L^\infty([0,T];\ L^2(\mathbb{R}^2_+))}\\
&+C\left\|\phi^3_m W^3_m f_m\right\|_{L^\infty([0,T];\ L^2(\mathbb{R}^2_+))}.
\end{aligned}
$$

The other terms

$$
\begin{aligned}
&\left\|\partial_y^2\Lambda^{-2/3}\phi^0_{m+1}W^0_{m+1}f_{m+1}\right\|_{L^\infty([0,T];\ L^2(\mathbb{R}^2_+))},\\
&\left\|\partial_y^3\Lambda^{-1}\phi^0_{m+1}W^0_{m+1}f_{m+1}\right\|_{L^\infty([0,T];\ L^2(\mathbb{R}^2_+))}
\end{aligned}
$$

can be treated in the same way, thanks to (4.5) and (4.6). So we omit it here. Thus the proof of Lemma 4.4 is complete. ∎

Proof of Lemma 2.5. Observe

$$
\begin{aligned}
(1+y)^{-\frac{\sigma}{2}} &= \left(\frac{(3m+\ell-1)\sigma}{2c}\right)^{-\frac{\sigma}{2}}\left(\frac{2c}{(3m+\ell-1)\sigma}+\frac{2cy}{(3m+\ell-1)\sigma}\right)^{-\frac{\sigma}{2}}\\
&\geq C\left(\frac{(3m+\ell-1)\sigma}{2c}\right)^{-\frac{\sigma}{2}}\left(1+\frac{2cy}{(3m+\ell-1)\sigma}\right)^{-\frac{\sigma}{2}}\\
&\geq Cm^{-\frac{\sigma}{2}}\left(1+\frac{2cy}{(3m+\ell-1)\sigma}\right)^{-\frac{\sigma}{2}}.
\end{aligned}
$$

Then

$$
(1+y)^{-\frac{\sigma}{2}}\left(1+\frac{2cy}{(3m+\ell-1)\sigma}\right)^{-\frac{(3m+\ell-1)\sigma}{2}}\geq Cm^{-\frac{\sigma}{2}}\left(1+\frac{2cy}{(3m+\ell-1)\sigma}\right)^{-\frac{(3m+\ell)\sigma}{2}}. \tag{4.11}
$$

Moreover, we find

$$\begin{aligned}&\left(1+\frac{2cy}{(3m+\ell-1)\sigma}\right)^{-\frac{(3m+\ell)\sigma}{2}}\\&\quad=\left(\frac{1}{(3m+\ell-1)\sigma}\right)^{-\frac{(3m+\ell)\sigma}{2}}((3m+\ell-1)\sigma+2cy)^{-\frac{(3m+\ell)\sigma}{2}}\\&\quad=\left(\frac{(3m+\ell)\sigma}{(3m+\ell-1)\sigma}\right)^{-\frac{(3m+\ell)\sigma}{2}}\left(\frac{(3m+\ell-1)}{(3m+\ell)}+\frac{2cy}{(3m+\ell)\sigma}\right)^{-\frac{(3m+\ell)\sigma}{2}}\\&\quad\geq\left(\frac{3m+\ell}{3m+\ell-1}\right)^{-\frac{(3m+\ell)\sigma}{2}}\left(1+\frac{2cy}{(3m+\ell)\sigma}\right)^{-\frac{(3m+\ell)\sigma}{2}}\\&\quad\geq C\left(1+\frac{2cy}{(3m+\ell)\sigma}\right)^{-\frac{(3m+\ell)\sigma}{2}},\end{aligned}$$

which along with (4.11) gives

$$\left(1+\frac{2cy}{(3m+\ell)\sigma}\right)^{-\frac{(3m+\ell)\sigma}{2}}\leq Cm^{\frac{\sigma}{2}}(1+y)^{-\frac{\sigma}{2}}\left(1+\frac{2cy}{(3m+\ell-1)\sigma}\right)^{-\frac{(3m+\ell-1)\sigma}{2}}.$$

As a result, recalling

$$(1+y)^{-\frac{\sigma}{2}}\Lambda^{1/3}W_m^{\ell-1}=(1+y)^{-\frac{\sigma}{2}}e^{2cy}\left(1+\frac{2cy}{(3m+\ell-1)\sigma}\right)^{-\frac{(3m+\ell-1)\sigma}{2}}(1+cy)^{-1}\Lambda^{\frac{\ell}{3}},$$

we have, observing $\phi^{-\frac12}\phi_m^\ell=\phi^{\frac12}\phi_m^{\ell-1}$,

$$\begin{aligned}&\|\phi^{-\frac12}\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\&\leq Cm^{\sigma/2}\big\|(1+y)^{-\frac{\sigma}{2}}\Lambda^{1/3}\phi^{\frac12}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)},\end{aligned}$$

that is, recalling $F=\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell f_m$ and $f=\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m$,

$$\|\phi^{-1/2}F\|_{L^2([0,T]\times\mathbb{R}_+^2)}\leq Cm^{\sigma/2}\big\|\langle y\rangle^{-\frac{\sigma}{2}}\Lambda^{1/3}f\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}.$$

Moreover, using (4.3) and (4.5) we have, observing $\phi_m^\ell\leq\phi^{1/2}\phi_m^{\ell-1}$,

$$\begin{aligned}&\|\partial_y^2\Lambda^{-2/3}F\|_{L^2([0,T]\times\mathbb{R}_+^2)}=\big\|\partial_y^2\Lambda^{-2/3}\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\&\leq C\big\|\partial_y^2\Lambda^{-1/3}\Lambda_\delta^{-2}\phi_m^\ell W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}+C\|\partial_y\Lambda^{-1/3}\Lambda_\delta^{-2}\phi_m^\ell W_m^{\ell-1}f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\&\quad+C\big\|\Lambda^{-1/3}\Lambda_\delta^{-2}\phi_m^\ell W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\&\leq C\big\|\partial_y^2\Lambda^{-1/3}f\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\&\quad+C\left(\|\phi_m^{\ell-1}W_m^{\ell-1}f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)}+\big\|\partial_y\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\right).\end{aligned}$$

Then combining the above inequalities, the first estimate in Lemma 2.5 follows. The second one can be deduced similarly. In fact, using (4.3) and (4.6) gives

$$\begin{aligned}
&\left\|\partial_y^3\Lambda^{-1}F\right\|_{L^2([0,T]\times\mathbb{R}_+^2)} = \left\|\partial_y^3\Lambda^{-1}\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell f_m\right\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
\le\ & C\left\|\partial_y^3\Lambda^{-2/3}\Lambda_\delta^{-2}\phi_m^\ell W_m^{\ell-1} f_m\right\|_{L^2([0,T]\times\mathbb{R}_+^2)} + C\left\|\partial_y^2\Lambda^{-2/3}\Lambda_\delta^{-2}\phi_m^\ell W_m^{\ell-1} f_m\right\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&+C\left\|\partial_y\Lambda^{-2/3}\Lambda_\delta^{-2}\phi_m^{\ell-1} W_m^{\ell-1} f_m\right\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&+C\left\|\Lambda^{-2/3}\Lambda_\delta^{-2}\phi_m^{\ell-1} W_m^{\ell-1} f_m\right\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
\le\ & C\left\|\partial_y^3\Lambda^{-2/3} f\right\|_{L^2([0,T]\times\mathbb{R}_+^2)} + C\left\|\partial_y^2\Lambda^{-1/3} f\right\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&+C\left(\|\phi_m^{\ell-1}W_m^{\ell-1}f_m\|_{L^2([0,T]\times\mathbb{R}_+^2)} + \left\|\partial_y\phi_m^{\ell-1}W_m^{\ell-1}f_m\right\|_{L^2([0,T]\times\mathbb{R}_+^2)}\right).
\end{aligned}$$

This is just the second estimate in Lemma 2.5. The proof is thus complete. ∎

5. Estimates of the nonlinear terms. In this section we estimate the nonlinear terms $\mathcal{Z}_{m,\ell,\delta}$ defined in (2.13), and prove the Proposition 2.3. Recall

$$\begin{aligned}
\mathcal{Z}_{m,\ell,\delta} =& -\sum_{j=1}^{m}\binom{m}{j}\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell(\partial_x^j u)f_{m+1-j} - \sum_{j=1}^{m-1}\binom{m}{j}\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell(\partial_x^j v)\partial_y f_{m-j}\\
&-\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]\sum_{j=1}^{m-1}\binom{m}{j}(\partial_x^j v)(\partial_x^{m-j}u)\\
&-2\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]f_m\\
&+\Lambda_\delta^{-2}\left(\partial_t\phi_m^\ell\right)W_m^\ell f_m + \left[u\partial_x + v\partial_y - \partial_y^2, \Lambda_\delta^{-2}\phi_m^\ell W_m^\ell\right]f_m\\
=\ & \mathcal{J}_{m,\ell,\delta} + \Lambda_\delta^{-2}\left(\partial_t\phi_m^\ell\right)W_m^\ell f_m + \left[u\partial_x + v\partial_y - \partial_y^2, \Lambda_\delta^{-2}\phi_m^\ell W_m^\ell\right]f_m,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{J}_{m,\ell,\delta} =& -\sum_{j=1}^{m}\binom{m}{j}\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell(\partial_x^j u)f_{m+1-j} - \sum_{j=1}^{m-1}\binom{m}{j}\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell(\partial_x^j v)\partial_y f_{m-j}\\
&-\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]\sum_{j=1}^{m-1}\binom{m}{j}(\partial_x^j v)(\partial_x^{m-j}u)\\
&-2\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]f_m.
\end{aligned}$$

We remark it suffices to prove the estimates (2.15) and (2.17) in Proposition 2.3, since the esimate (2.16) can be treated exactly the same as (2.15). Next we will proceed to prove (2.15) and (2.17) through the following Propositions 5.1 and 5.2. Proposition 5.2 is devoted to treating the term $\mathcal{J}_{m,\ell,\delta}$ in the definition of $\mathcal{Z}_{m,\ell,\delta}$, while the other two terms are estimated in Proposition 5.1.

To simplify the notations, we will use C to denote different constants depending only on σ, c, and the constants C_0, C_* in Theorem 1.1, but independent of m and δ.

PROPOSITION 5.1. *We have, denoting* $F = \Lambda_\delta^{-2}\phi_m^\ell W_m^\ell f_m$ *and* $\tilde f = \phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m$,

$$\begin{aligned}&\big\|\phi^{1/2}\Lambda_\delta^{-2}\left(\partial_t\phi_m^\ell\right)W_m^\ell f_m\big\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\&+\big\|\phi^{1/2}[u\partial_x+v\partial_y-\partial_y^2,\ \Lambda_\delta^{-2}\phi_m^\ell W_m^\ell]f_m\big\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\\le{}& m\,C\big\|\phi^{-1/2}F\big\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}+C\big\|\partial_yF\big\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\end{aligned}$$

and

$$\begin{aligned}&\big\|\Lambda^{-\frac23}\partial_y\phi^{1/2}\Lambda_\delta^{-2}\left(\partial_t\phi_m^{\ell-1}\right)W_m^{\ell-1}f_m\big\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\&\qquad+\big\|\Lambda^{-\frac23}\partial_y\phi^{1/2}[u\partial_x+v\partial_y-\partial_y^2,\ \Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}]f_m\big\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\&\le C\big\|\left\langle y\right\rangle^{-\sigma}\Lambda^{\frac13}\tilde f\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}+C\big\|\partial_y^2\Lambda^{-\frac23}\tilde f\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\&\quad+mC\big\|\Lambda^{-\frac23}\phi^{-1/2}\partial_y\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}+C\big\|\Lambda^{-\frac23}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}.\end{aligned}$$

Proof. It is sufficient to prove the second estimate in Proposition 5.1, since the treatment of the first one is similar and easier and we omit it here for brevity. Observe

$$\left|\partial_t\phi_m^{\ell-1}\right|\le 3m\phi_m^{\ell-2}\le 3m\phi_m^{\ell-1}\phi^{-1},$$

and thus

$$\begin{aligned}&\big\|\Lambda^{-\frac23}\partial_y\phi^{1/2}\Lambda_\delta^{-2}\left(\partial_t\phi_m^{\ell-1}\right)W_m^{\ell-1}f_m\big\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\&\le 3m\big\|\Lambda^{-\frac23}\phi^{-1/2}\partial_y\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}.\end{aligned}\tag{5.1}$$

We write, using (3.1),

$$\begin{aligned}&\big\|\Lambda^{-\frac23}\partial_y\phi^{1/2}[u\partial_x+v\partial_y-\partial_y^2,\ \Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}]f_m\big\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\\le{}&\big\|\left[u\partial_x+v\partial_y-\partial_y^2,\ \Lambda^{-\frac23}\partial_y\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}\right]f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\&+\big\|\left[u\partial_x+v\partial_y-\partial_y^2,\ \Lambda^{-\frac23}\partial_y\phi^{1/2}\right]\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\\overset{\text{def}}{=}{}&Q_{5,1}+Q_{5,2}.\end{aligned}$$

We first estimate $Q_{5,1}$. Observe

$$\begin{aligned}&\big\|\left[u\partial_x,\ \Lambda^{-\frac23}\partial_y\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}\right]f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\\le{}&\big\|\left[u\partial_x,\ \Lambda^{-\frac23}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}\partial_y\right]\phi^{1/2}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\&+\big\|\Big[u\partial_x,\ \Lambda^{-\frac23}\Lambda_\delta^{-2}\phi_m^{\ell-1}\left[\partial_y,\ W_m^{\ell-1}\right]\Big]\phi^{1/2}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}.\end{aligned}$$

On the other hand, we compute, using Lemma 3.1 and (4.4),

$$
\begin{aligned}
&\big\|\big[u\partial_x,\ \Lambda^{-\frac{2}{3}}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}\partial_y\big]\phi^{1/2}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&\leq \big\|\big[u\partial_x,\ \Lambda^{-\frac{2}{3}}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}\big]\partial_y\phi^{1/2}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&\quad+\big\|\Lambda^{-\frac{2}{3}}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}\big[u\partial_x,\ \partial_y\big]\phi^{1/2}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&\leq C\big\|\Lambda^{-\frac{2}{3}}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}\partial_y\phi^{1/2}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&\quad+\big\|\Lambda^{-\frac{2}{3}}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}(\partial_y u)\partial_x\phi^{1/2}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&\leq C\big\|\partial_y\Lambda^{-\frac{2}{3}}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}\phi^{1/2}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&\quad+C\big\|\Lambda^{-\frac{2}{3}}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}\phi^{1/2}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&\quad+\big\|(\partial_y u)\Lambda^{-\frac{2}{3}}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}\partial_x\phi^{1/2}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&\quad+\big\|\big[\Lambda^{-\frac{2}{3}}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1},\ (\partial_y u)\big]\partial_x\phi^{1/2}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&\leq C\big\|\partial_y\Lambda^{-\frac{2}{3}}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}+C\big\|\Lambda^{-\frac{2}{3}}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&\quad+C\big\|\left\langle y\right\rangle^{-\sigma}\Lambda^{\frac{1}{3}}\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}.
\end{aligned}
$$

Similarly we also have, using again Lemma 3.1,

$$
\begin{aligned}
&\Big\|\Big[u\partial_x,\ \Lambda^{-\frac{2}{3}}\Lambda_\delta^{-2}\phi_m^{\ell-1}\big[\partial_y,\ W_m^{\ell-1}\big]\ \Big]\phi^{1/2}f_m\Big\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
&\quad\leq C\big\|\Lambda^{-\frac{2}{3}}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}.
\end{aligned}
$$

As a result, combining these inequalities, we have

$$
\begin{aligned}
&\big\|\big[u\partial_x,\ \Lambda^{-\frac{2}{3}}\partial_y\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}\big]f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&\leq C\big\|\partial_y\Lambda^{-\frac{2}{3}}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}+C\big\|\Lambda^{-\frac{2}{3}}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&\quad+C\big\|\left\langle y\right\rangle^{-\sigma}\Lambda^{\frac{1}{3}}\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}.
\end{aligned}
$$

Similarly, repeating the above arguments with $u\partial_x$ replaced by $v\partial_y$ and ∂_y^2, respectively, one has

$$
\begin{aligned}
&\big\|\big[v\partial_y,\ \Lambda^{-\frac{2}{3}}\partial_y\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}\big]f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&\leq C\big\|\partial_y^2\Lambda^{-\frac{2}{3}}\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}+C\big\|\Lambda^{-\frac{2}{3}}\partial_y\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&\quad+C\big\|\Lambda^{-\frac{2}{3}}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\big\|\big[\partial_y^2,\ \Lambda^{-\frac{2}{3}}\partial_y\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}\big]f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&\leq C\big\|\partial_y^2\Lambda^{-\frac{2}{3}}\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}+C\big\|\Lambda^{-\frac{2}{3}}\partial_y\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&\quad+C\big\|\Lambda^{-\frac{2}{3}}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}.
\end{aligned}
$$

As a result, we conclude, combining these inequalities,

$$\begin{aligned}Q_{5,1} &= \big\|\big[u\partial_x + v\partial_y - \partial_y^2,\ \Lambda^{-\frac{2}{3}}\partial_y\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}\big]f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&\le C\big\|\,\langle y\rangle^{-\sigma}\,\Lambda^{\frac{1}{3}}\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&\quad+C\big\|\partial_y^2\Lambda^{-\frac{2}{3}}\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&\quad+C\big\|\partial_y\Lambda^{-\frac{2}{3}}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}+C\big\|\Lambda^{-\frac{2}{3}}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}.\end{aligned}$$

The term $Q_{5,2}$ can be treated similarly and easily, and we have

$$\begin{aligned}Q_{5,2} &= \big\|\big[u\partial_x + v\partial_y - \partial_y^2,\ \Lambda^{-\frac{2}{3}}\partial_y\phi^{1/2}\big]\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&\le C\big\|\,\langle y\rangle^{-\sigma}\,\Lambda^{\frac{1}{3}}\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&\quad+C\big\|\partial_y^2\Lambda^{-\frac{2}{3}}\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&\quad+C\big\|\partial_y\Lambda^{-\frac{2}{3}}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}+C\big\|\Lambda^{-\frac{2}{3}}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}.\end{aligned}$$

Thus

$$\begin{aligned}&\big\|\Lambda^{-\frac{2}{3}}\partial_y\phi^{1/2}[u\partial_x + v\partial_y - \partial_y^2,\ \Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}]f_m\big\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
&\le C\big\|\,\langle y\rangle^{-\sigma}\,\Lambda^{\frac{1}{3}}\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&\quad+C\big\|\partial_y^2\Lambda^{-\frac{2}{3}}\phi^{1/2}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}\\
&\quad+C\big\|\partial_y\Lambda^{-\frac{2}{3}}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}+C\big\|\Lambda^{-\frac{2}{3}}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2([0,T]\times\mathbb{R}_+^2)}.\end{aligned}$$

This along with (5.1) gives the second estimate in Proposition 5.1. The proof is thus complete. ∎

PROPOSITION 5.2. *Under the induction hypothesis* (2.9), (2.10), *we have, denoting* $F = \Lambda_\delta^{-2}\phi_m^\ell W_m^\ell f_m$,

$$\begin{aligned}&\big\|\phi^{1/2}\mathcal{J}_{m,\ell,\delta}\big\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
&\le mC\big\|F\big\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
&\quad+CA^{m-6}\left((m-5)!\right)^{3(1+\sigma)},\end{aligned}$$

and

$$\begin{aligned}&\big\|\Lambda^{-2/3}\partial_y\phi^{1/2}\mathcal{J}_{m,\ell-1,\delta}\big\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
&\le mC\left(\big\|\Lambda^{-2/3}\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\right.\\
&\quad\left.+\big\|\Lambda^{-2/3}\partial_y\Lambda_\delta^{-2}\phi_m^{\ell-1}W_m^{\ell-1}f_m\big\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\right)\\
&\quad+C\,A^{m-6}\left((m-5)!\right)^{3(1+\sigma)},\end{aligned}$$

where the constant $C>0$ *is independent of* m *and* $\delta>0$.

We first prove the first estimate in Proposition 5.2. In view of the definition given at the beginning of this section, we see that

$$
\begin{aligned}
&\big\|\phi^{1/2}\mathcal{J}_{m,\ell,\delta}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)} \leq \big\|\mathcal{J}_{m,\ell,\delta}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&\leq \sum_{j=1}^{m}\binom{m}{j}\big\|\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell(\partial_x^j u)f_{m+1-j}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&\quad+\sum_{j=1}^{m-1}\binom{m}{j}\big\|\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell(\partial_x^j v)\partial_y f_{m-j}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&\quad+\sum_{j=1}^{m-1}\binom{m}{j}\big\|\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell\left[\partial_y\left(\partial_y\omega/\omega\right)\right](\partial_x^j v)(\partial_x^{m-j}u)\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&\quad+2\big\|\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell\left[\partial_y\left(\partial_y\omega/\omega\right)\right]f_m\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}.
\end{aligned}
\tag{5.2}
$$

And we will proceed to estimate each term on the right-hand side of (5.2), and state the results as the following three lemmas.

Lemma 5.3. *Under the same assumption as in Proposition* 2.3*, we have*

$$
\begin{aligned}
&\sum_{j=1}^{m-1}\binom{m}{j}\big\|\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell(\partial_x^j v)\partial_y f_{m-j}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&\leq mC\big\|\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell f_m\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}+CA^{m-6}\left((m-5)!\right)^{3(1+\sigma)}.
\end{aligned}
$$

Proof. We first split the summation as follows:

$$
\begin{aligned}
&\sum_{j=1}^{m-1}\binom{m}{j}\big\|\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell(\partial_x^j v)\partial_y f_{m-j}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&=\sum_{j=m-2}^{m-1}\binom{m}{j}\big\|\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell(\partial_x^j v)\partial_y f_{m-j}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&\quad+\sum_{j=1}^{m-3}\binom{m}{j}\big\|\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell(\partial_x^j v)\partial_y f_{m-j}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}.
\end{aligned}
$$

Moreover, as for the last term on the right-hand side, we use (4.3) to compute

$$
\begin{aligned}
&\big\|\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell(\partial_x^j v)\partial_y f_{m-j}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&\leq \big\|\phi_m^\ell W_m^0\Lambda^{\ell/3}(\partial_x^j v)\partial_y f_{m-j}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&\leq \big\|\phi_m^\ell W_m^0(\partial_x^j v)\partial_y f_{m-j}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}+\big\|\phi_m^\ell W_m^0\partial_x(\partial_x^j v)\partial_y f_{m-j}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&\leq \big\|\phi_m^\ell W_m^0(\partial_x^j v)\partial_y f_{m-j}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}+\big\|\phi_m^\ell W_m^0(\partial_x^{j+1} v)\partial_y f_{m-j}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&\quad+\big\|\phi_m^\ell W_m^0(\partial_x^j v)(\partial_y\partial_x f_{m-j})\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&\sum_{j=1}^{m-1}\binom{m}{j}\left\|\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell(\partial_x^j v)\partial_y f_{m-j}\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
&\leq \sum_{j=m-2}^{m-1}\binom{m}{j}\left\|\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell(\partial_x^j v)\partial_y f_{m-j}\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
&\quad+\sum_{j=1}^{m-3}\binom{m}{j}\left\|\phi_m^\ell W_m^0(\partial_x^j v)\partial_y f_{m-j}\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
&\quad+\sum_{j=1}^{m-3}\binom{m}{j}\left\|\phi_m^\ell W_m^0(\partial_x^{j+1} v)\partial_y f_{m-j}\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
&\quad+\sum_{j=1}^{m-3}\binom{m}{j}\left\|\phi_m^\ell W_m^0(\partial_x^j v)(\partial_y\partial_x f_{m-j})\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}.
\end{aligned}\tag{5.3}
$$

Next we estimate step by step the terms on the right side of (5.3).

(a) We treat in this step the first term on the right-hand side of (5.3), and prove that

$$
\begin{aligned}
&\sum_{j=m-2}^{m-1}\binom{m}{j}\left\|\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell(\partial_x^j v)\partial_y f_{m-j}\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
&\leq m\,C\left\|\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}+CA^{m-6}\left((m-5)!\right)^{3(1+\sigma)}.
\end{aligned}\tag{5.4}
$$

To do so, direct computation gives

$$
\begin{aligned}
&\sum_{j=m-2}^{m-1}\binom{m}{j}\left\|\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell(\partial_x^j v)\partial_y f_{m-j}\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
&\leq \sum_{j=m-2}^{m-1}\binom{m}{j}\left\|\Lambda_\delta^{-2}\Lambda^{\ell/3}e^{2cy}\left(1+cy\right)^{-1}\phi_m^\ell(\partial_x^j v)\partial_y f_{m-j}\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
&\leq \sum_{j=m-2}^{m-1}\binom{m}{j}\left\|e^{2cy}(\partial_y f_{m-j})\Lambda_\delta^{-2}\Lambda^{\ell/3}\left(1+cy\right)^{-1}\phi_m^\ell\partial_x^j v\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
&\quad+\sum_{j=m-2}^{m-1}\binom{m}{j}\left\|\left[e^{2cy}(\partial_y f_{m-j}),\ \Lambda_\delta^{-2}\Lambda^{\ell/3}\right]\left(1+cy\right)^{-1}\phi_m^\ell\partial_x^j v\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}.
\end{aligned}
$$

On the other hand, by (2.7),

$$
\begin{aligned}
&\sum_{j=m-2}^{m-1}\binom{m}{j}\left\|e^{2cy}(\partial_y f_{m-j})\Lambda_\delta^{-2}\Lambda^{\ell/3}\left(1+cy\right)^{-1}\phi_m^\ell\partial_x^j v\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
&\leq C\sum_{j=m-2}^{m-1}\binom{m}{j}\left\|(1+cy)^{-1}\right\|_{L^2(\mathbb{R}_+;\ L^\infty([0,T]\times\mathbb{R}_x))}\left\|\Lambda_\delta^{-2}\phi_m^\ell\Lambda^{\ell/3}\partial_x^j v\right\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}\\
&\leq C\sum_{j=m-2}^{m-1}\binom{m}{j}\left\|\Lambda_\delta^{-2}\phi_m^\ell\Lambda^{\ell/3}\partial_x^j v\right\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}.
\end{aligned}
$$

Similarly, we have, by virtue of Lemma 3.1,

$$\begin{aligned}
&\sum_{j=m-2}^{m-1}\binom{m}{j}\left\|\left[e^{2cy}(\partial_y f_{m-j}),\ \Lambda_\delta^{-2}\Lambda^{\ell/3}\right](1+cy)^{-1}\,\phi_m^\ell\partial_x^j v\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
&\leq C\sum_{j=m-2}^{m-1}\binom{m}{j}\left\|(1+cy)^{-1}\,\phi_m^\ell\partial_x^j v\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
&\leq C\sum_{j=m-2}^{m-1}\binom{m}{j}\left\|\phi_m^\ell\Lambda^{\ell/3}\partial_x^j v\right\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}.
\end{aligned}$$

Thus combining these inequalities, we obtain

$$\begin{aligned}
&\sum_{j=m-2}^{m-1}\binom{m}{j}\left\|\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell(\partial_x^j v)\partial_y f_{m-j}\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
&\leq C\sum_{j=m-2}^{m-1}\binom{m}{j}\left\|\Lambda_\delta^{-2}\phi_m^\ell\Lambda^{\ell/3}\partial_x^j v\right\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}\\
&\leq Cm\left\|\Lambda_\delta^{-2}\phi_m^\ell\Lambda^{\ell/3}\partial_x^{m-1}v\right\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}\\
&\quad+Cm^2\left\|\Lambda_\delta^{-2}\phi_m^\ell\Lambda^{\ell/3}\partial_x^{m-2}v\right\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}.
\end{aligned}$$

Moreover, observe

$$\begin{aligned}
&\left\|\Lambda_\delta^{-2}\phi_m^\ell\Lambda^{\ell/3}\partial_x^{m-2}v\right\|^2_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}\\
&\leq\left\|\Lambda_\delta^{-2}\phi_m^\ell\Lambda^{\ell/3}\partial_x^{m-1}v\right\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}\left\|\Lambda_\delta^{-2}\phi_m^\ell\Lambda^{\ell/3}\partial_x^{m-3}v\right\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))},
\end{aligned}$$

and thus

$$\begin{aligned}
&m^2\left\|\Lambda_\delta^{-2}\phi_m^\ell\Lambda^{\ell/3}\partial_x^{m-2}v\right\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}\\
&\leq m\left\|\Lambda_\delta^{-2}\phi_m^\ell\Lambda^{\ell/3}\partial_x^{m-1}v\right\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}\\
&\quad+m^3\left\|\Lambda_\delta^{-2}\phi_m^\ell\Lambda^{\ell/3}\partial_x^{m-3}v\right\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}\\
&\leq m\left\|\Lambda_\delta^{-2}\phi_m^\ell\Lambda^{\ell/3}\partial_x^{m-1}v\right\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}\\
&\quad+m^3\left\|\Lambda_\delta^{-2}\phi_m^\ell\partial_x^{m-3}v\right\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}\\
&\quad+m^3\left\|\Lambda_\delta^{-2}\phi_m^\ell\partial_x^{m-2}v\right\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}\\
&\leq m\left\|\Lambda_\delta^{-2}\phi_m^\ell\Lambda^{\ell/3}\partial_x^{m-1}v\right\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}\\
&\quad+m^3\left\|\phi_{m-2}^0\partial_x^{m-3}v\right\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}\\
&\quad+m^3\left\|\phi_{m-1}^0\partial_x^{m-2}v\right\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}.
\end{aligned}$$

Then we have, combining the above inequalities,

$$\begin{aligned}
&\sum_{j=m-2}^{m-1}\binom{m}{j}\left\|e^{2cy}(\partial_y f_{m-j})\Lambda_\delta^{-2}\Lambda^{\ell/3}\,(1+cy)^{-1}\,\phi_m^\ell\partial_x^j v\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
&\leq Cm\left\|\Lambda_\delta^{-2}\phi_m^\ell\Lambda^{\ell/3}\partial_x^{m-1}v\right\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}\\
&\quad+Cm^3\left\|\phi_{m-2}^0\partial_x^{m-3}v\right\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}
\end{aligned}$$

$$
\begin{aligned}
&+ Cm^3\big\|\phi^0_{m-1}\partial_x^{m-2}v\big\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}\\
&\le Cm\big\|\Lambda_\delta^{-2}\phi^\ell_m W^\ell_m f_m\big\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}\\
&\quad + Cm^3\big\|\phi^0_{m-2}W^0_{m-2}f_{m-2}\big\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}\\
&\quad + Cm^3\big\|\phi^0_{m-1}W^0_{m-1}f_{m-1}\big\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))},
\end{aligned}
$$

the last inequality following from (4.8). This, along with the estimate

$$
\begin{aligned}
&m^3\big\|\phi^0_{m-2}W^0_{m-2}f_{m-2}\big\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}\\
&+m^3\big\|\phi^0_{m-1}W^0_{m-1}f_{m-1}\big\|_{L^\infty(\mathbb{R}_+;\ L^2([0,T]\times\mathbb{R}_x))}\\
&\le CA^{m-6}\left((m-5)!\right)^{3(1+\sigma)}
\end{aligned}
$$

due to the inductive assumption (2.9), gives the desired estimate (5.4).

(b) We will estimate in this step the second and the third terms on the right-hand side of (5.3), and prove that

$$
\begin{aligned}
&\sum_{j=1}^{m-3}\binom{m}{j}\big\|\phi^\ell_m W^0_m(\partial_x^j v)\partial_y f_{m-j}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&+\sum_{j=1}^{m-3}\binom{m}{j}\big\|\phi^\ell_m W^0_m(\partial_x^{j+1} v)\partial_y f_{m-j}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&\le CA^{m-6}\left((m-5)!\right)^{3(1+\sigma)}.
\end{aligned}
\tag{5.5}
$$

For this purpose we write, denoting by $[m/2]$ the largest integer less than or equal to $m/2$,

$$
\begin{aligned}
&\sum_{j=1}^{m-3}\binom{m}{j}\big\|\phi^\ell_m W^0_m(\partial_x^{j+1} v)\partial_y f_{m-j}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&\le \sum_{j=1}^{[m/2]}\binom{m}{j}\big\|\phi^\ell_m W^0_m(\partial_x^{j+1} v)(\partial_y f_{m-j})\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&\quad+\sum_{j=[m/2]+1}^{m-3}\binom{m}{j}\big\|\phi^\ell_m W^0_m(\partial_x^{j+1} v)\partial_y f_{m-j}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&= S_1+S_2.
\end{aligned}
\tag{5.6}
$$

We first treat S_1. Using the inequalities

$$
\phi^\ell_m\le\phi^0_m\le\phi^0_{j+3}\phi^0_{m-j},\quad W^0_m\le W^0_{m-j}\ \text{for}\ j\ge 1,
$$

gives

$$
\begin{aligned}
S_1&=\sum_{j=1}^{[m/2]}\binom{m}{j}\big\|\phi^\ell_m W^0_m(\partial_x^{j+1} v)\partial_y f_{m-j}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&\le\sum_{j=1}^{[m/2]}\binom{m}{j}\big\|\phi^0_{j+3}\partial_x^{j+1}v\big\|_{L^\infty\left([0,T]\times\mathbb{R}^2_+\right)}\big\|\phi^0_{m-j}W^0_{m-j}\partial_y f_{m-j}\big\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}.
\end{aligned}
\tag{5.7}
$$

By the Sobolev inequality, we have

$$
\begin{aligned}
&\left\|\phi_{j+3}^0\partial_x^{j+1}v\right\|_{L^\infty\left([0,T]\times\mathbb{R}_+^2\right)}\\
&\leq C\left\|\phi_{j+3}^0\partial_x^{j+1}v\right\|_{L^\infty([0,T]\times\mathbb{R}_+;\ L^2(\mathbb{R}_x))}+C\left\|\phi_{j+3}^0\partial_x^{j+2}v\right\|_{L^\infty([0,T]\times\mathbb{R}_+;\ L^2(\mathbb{R}_x))}\\
&\leq C\left\|\phi_{j+2}^0\partial_x^{j+1}v\right\|_{L^\infty([0,T]\times\mathbb{R}_+;\ L^2(\mathbb{R}_x))}+C\left\|\phi_{j+3}^0\partial_x^{j+2}v\right\|_{L^\infty([0,T]\times\mathbb{R}_+;\ L^2(\mathbb{R}_x))}\\
&\leq C\left\|\phi_{j+2}^0W_{j+2}^0f_{j+2}\right\|_{L^\infty\left([0,T];\ L^2(\mathbb{R}_+^2)\right)}+C\left\|\phi_{j+3}^0W_{j+3}^0f_{j+3}\right\|_{L^\infty\left([0,T];\ L^2(\mathbb{R}_+^2)\right)},
\end{aligned}
$$

the second inequality using (4.1) and the last inequality following from (4.8). As a result, we use the hypothesis of induction (2.9) and the initial hypothesis of induction (2.7) to conclude that if $4\leq j\leq[m/2]$ then

$$
\begin{aligned}
&\left\|\phi_{j+3}^0\partial_x^{j+1}v\right\|_{L^\infty\left([0,T]\times\mathbb{R}_+^2\right)}\\
&\leq C\left(A^{j-3}\left((j-3)!\right)^{3(1+\sigma)}+A^{j-2}\left((j-2)!\right)^{3(1+\sigma)}\right)\\
&\leq CA^{j-2}\left((j-2)!\right)^{3(1+\sigma)},
\end{aligned}
$$

and if $1\leq j\leq 3$

$$
\left\|\phi_{j+3}^0\partial_x^{j+1}v\right\|_{L^\infty\left([0,T]\times\mathbb{R}_+^2\right)}\leq C.
$$

Moreover, using (4.4) and also the inductive assumption (2.9), we calculate, for any $1\leq j\leq[m/2]$,

$$
\begin{aligned}
&\left\|\phi_{m-j}^0W_{m-j}^0\partial_yf_{m-j}\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
&\leq\left\|\partial_y\phi_{m-j}^0W_{m-j}^0f_{m-j}\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}+\left\|\phi_{m-j}^0\left[\partial_y,\ W_{m-j}^0\right]f_{m-j}\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\
&\leq\left\|\partial_y\phi_{m-j}^0W_{m-j}^0f_{m-j}\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}+C\left\|\phi_{m-j}^0W_{m-j}^0f_{m-j}\right\|_{L^\infty\left([0,T],\ L^2(\mathbb{R}_+^2)\right)}\\
&\leq CA^{m-j-5}\left((m-j-5)!\right)^{3(1+\sigma)}.
\end{aligned}
$$

Putting these inequalities into (5.7) gives

$$
\begin{aligned}
S_1&\leq C\sum_{j=4}^{[m/2]}\frac{m!}{j!(m-j)!}A^{j-2}\left((j-2)!\right)^{3(1+\sigma)}\left(A^{m-j-5}\left((m-j-5)!\right)^{3(1+\sigma)}\right)\\
&\quad+C\sum_{j=1}^{3}\frac{m!}{j!(m-j)!}\left(A^{m-j-5}\left((m-j-5)!\right)^{3(1+\sigma)}\right)\\
&\leq C\sum_{j=4}^{[m/2]}\frac{m!}{j^2(m-j)^5}A^{m-7}\left((j-2)!\right)^{3(1+\sigma)-1}\left((m-j-5)!\right)^{3(1+\sigma)-1}\\
&\quad+CA^{m-6}\left((m-5)!\right)^{3(1+\sigma)}\\
&\leq C\sum_{j=4}^{[m/2]}\frac{m!}{j^2(m/2)^5}A^{m-7}\left((m-7)!\right)^{3(1+\sigma)-1}+CA^{m-6}\left((m-5)!\right)^{3(1+\sigma)}\\
&\leq C(m-5)!A^{m-7}\left((m-7)!\right)^{3(1+\sigma)-1}+CA^{m-6}\left((m-5)!\right)^{3(1+\sigma)}\\
&\leq CA^{m-6}\left((m-5)!\right)^{3(1+\sigma)}.
\end{aligned}
\tag{5.8}
$$

We now treat S_2. Using the inequalities

$$\phi_m^\ell \le \phi_m^0 \le \phi_{j+2}^0 \phi_{m-j+1}^0, \quad W_m^0 \le W_{m-j+1}^0 \text{ for } j \ge 1,$$

and thus

$$\begin{aligned} S_2 &= \sum_{j=[m/2]+1}^{m-3} \binom{m}{j} \left\| \phi_m^\ell W_m^0 (\partial_x^{j+1} v) \partial_y f_{m-j} \right\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ &\le \sum_{j=[\frac{m}{2}]+1}^{m-3} \binom{m}{j} \left\| \phi_{j+2}^0 \partial_x^{j+1} v \right\|_{L^\infty([0,T]\times\mathbb{R}_+;\ L^2(\mathbb{R}_x))} \\ &\qquad\qquad \times \left\| \phi_{m-j+1}^0 W_{m-j+1}^0 \partial_y f_{m-j} \right\|_{L^2([0,T]\times\mathbb{R}_+;\ L^\infty(\mathbb{R}_x))} \\ &\le \sum_{j=[m/2]+1}^{m-3} \binom{m}{j} \left\| \phi_{j+2}^0 W_{j+2}^0 f_{j+2} \right\|_{L^\infty([0,T];\ L^2(\mathbb{R}_+^2))} \\ &\qquad\qquad \times \left\| \phi_{m-j+1}^0 W_{m-j+1}^0 \partial_y f_{m-j} \right\|_{L^2([0,T]\times\mathbb{R}_+;\ L^\infty(\mathbb{R}_x))}, \end{aligned} \tag{5.9}$$

the last inequality using (4.8). As for the last factor in the above inequality, we use the Sobolev inequality, (4.1), and (4.2) to compute

$$\begin{aligned} &\left\| \phi_{m-j+1}^0 W_{m-j+1}^0 \partial_y f_{m-j} \right\|_{L^2([0,T]\times\mathbb{R}_+;\ L^\infty(\mathbb{R}_x))} \\ &\le C \left\| \phi_{m-j+1}^0 W_{m-j+1}^0 \partial_y f_{m-j} \right\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ &+ C \left\| \phi_{m-j+1}^0 W_{m-j+1}^0 \partial_y \partial_x f_{m-j} \right\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ &\le C \left\| \phi_{m-j}^0 W_{m-j}^0 \partial_y f_{m-j} \right\|_{L^2([0,T]\times\mathbb{R}_+^2)} + C \left\| \phi_{m-j+1}^0 W_{m-j+1}^0 \partial_y \partial_x f_{m-j} \right\|_{L^2([0,T]\times\mathbb{R}_+^2)}. \end{aligned}$$

On the other hand, in view of the definition of f_m, we have

$$\begin{aligned} &\left\| \phi_{m-j+1}^0 W_{m-j+1}^0 \partial_y \partial_x f_{m-j} \right\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ &\le \left\| \phi_{m-j+1}^0 W_{m-j+1}^0 \partial_y f_{m-j+1} \right\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ &\quad + \left\| \phi_{m-j+1}^0 W_{m-j+1}^0 \partial_y (\partial_x^{m-j} u) \partial_x \left((\partial_y \omega)/\omega \right) \right\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ &\le \left\| \phi_{m-j+1}^0 W_{m-j+1}^0 \partial_y f_{m-j+1} \right\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ &\quad + \left\| \phi_{m-j+1}^0 W_{m-j+1}^0 (\partial_x^{m-j} \omega) \partial_x \left((\partial_y \omega)/\omega \right) \right\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ &\quad + \left\| \phi_{m-j+1}^0 W_{m-j+1}^0 (\partial_x^{m-j} u) \partial_x \partial_y \left((\partial_y \omega)/\omega \right) \right\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ &\le C \left\| \phi_{m-j+1}^0 W_{m-j+1}^0 \partial_y f_{m-j+1} \right\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ &\quad + C \left\| \langle y \rangle^{-1} \phi_{m-j}^0 W_{m-j}^0 \partial_x^{m-j} \omega \right\|_{L^2([0,T]\times\mathbb{R}_+^2)} \\ &\quad + C \left\| \langle y \rangle^{-1} \phi_{m-j}^0 W_{m-j}^0 \partial_x^{m-j} u \right\|_{L^2([0,T]\times\mathbb{R}_+^2)}, \end{aligned}$$

the last inequality using (4.1) and (4.2). Combining these inequalities, we conclude

$$\begin{aligned}
&\left\|\phi^0_{m-j+1}W^0_{m-j+1}\partial_y f_{m-j}\right\|_{L^2([0,T]\times\mathbb{R}_+;\ L^\infty(\mathbb{R}_x))}\\
&\leq C\left\|\phi^0_{m-j}W^0_{m-j}\partial_y f_{m-j}\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}+C\left\|\phi^0_{m-j+1}W^0_{m-j+1}\partial_y f_{m-j+1}\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&\quad+C\left\|\langle y\rangle^{-1}\phi^0_{m-j}W^0_{m-j}\partial_x^{m-j}\omega\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&\quad+C\left\|\langle y\rangle^{-1}\phi^0_{m-j}W^0_{m-j}\partial_x^{m-j}u\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&\leq C\left\|\partial_y\phi^0_{m-j}W^0_{m-j}f_{m-j}\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}+C\left\|\partial_y\phi^0_{m-j+1}W^0_{m-j+1}f_{m-j+1}\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}\\
&\quad+C\left\|\phi^0_{m-j}W^0_{m-j}f_{m-j}\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)}+C\left\|\phi^0_{m-j+1}W^0_{m-j+1}f_{m-j+1}\right\|_{L^2\left([0,T]\times\mathbb{R}^2_+\right)},
\end{aligned}$$

where the last inequality follows from (4.9) and (4.4). This, along with the inductive assumptions (2.9), yields, if $\left[m/2\right]+1\leq j\leq m-4$, then

$$\begin{aligned}
&\left\|\phi^0_{m-j+1}W^0_{m-j+1}\partial_y f_{m-j}\right\|_{L^2([0,T]\times\mathbb{R}_+;\ L^\infty(\mathbb{R}_x))}\\
&\leq CA^{m-j-5}\left((m-j-5)!\right)^{3(1+\sigma)}+CA^{m-j-4}\left((m-j-4)!\right)^{3(1+\sigma)}\\
&\leq CA^{m-j-4}\left((m-j-4)!\right)^{3(1+\sigma)},
\end{aligned}$$

and if $j=m-3$, then

$$\left\|\phi^0_{m-j+1}W^0_{m-j+1}\partial_y f_{m-j}\right\|_{L^2([0,T]\times\mathbb{R}_+;\ L^\infty(\mathbb{R}_x))}\leq C$$

due to the initial hypothesis of induction (2.7). On the other hand, the inductive assumptions (2.9) yield, for any $\left[m/2\right]+1\leq j\leq m-3$,

$$\left\|\phi^0_{j+2}W^0_{j+2}f_{j+2}\right\|_{L^\infty\left([0,T];\ L^2(\mathbb{R}^2_+)\right)}\leq A^{j-3}\left((j-3)!\right)^{3(1+\sigma)}.$$

Putting these estimates into (5.9), we have

$$\begin{aligned}
S_2&\leq C\sum_{j=\left[m/2\right]+1}^{m-4}\frac{m!}{j!(m-j)!}A^{j-3}\big((j-3)!\big)^{3(1+\sigma)}\Big(A^{m-j-4}\left((m-j-4)!\right)^{3(1+\sigma)}\Big)\\
&\quad+C\sum_{j=m-3}^{m-3}\frac{m!}{j!(m-j)!}A^{j-3}\big((j-3)!\big)^{3(1+\sigma)}\\
&\leq C\sum_{j=\left[m/2\right]+1}^{m-4}\frac{m!}{j^3(m-j)^4}A^{m-7}\big((j-3)!\big)^{3(1+\sigma)-1}\left((m-j-4)!\right)^{3(1+\sigma)-1}\\
&\quad+CA^{m-6}\big((m-5)!\big)^{3(1+\sigma)}\\
&\leq C\sum_{j=\left[m/2\right]+1}^{m-4}\frac{m!}{(m/2)^3(m-j)^4}A^{m-7}\left((m-7)!\right)^{3(1+\sigma)-1}+CA^{m-6}\big((m-5)!\big)^{3(1+\sigma)}\\
&\leq C(m-3)!A^{m-7}\left((m-7)!\right)^{3(1+\sigma)-1}+CA^{m-6}\big((m-5)!\big)^{3(1+\sigma)}\\
&\leq CA^{m-6}\left((m-5)!\right)^{3(1+\sigma)}.
\end{aligned}$$

This along with (5.8) and (5.6) yields

$$\sum_{j=1}^{m-3}\binom{m}{j}\left\|\phi_m^\ell W_m^0(\partial_x^{j+1}v)\partial_y f_{m-j}\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\leq CA^{m-6}\left((m-5)!\right)^{3(1+\sigma)}.$$

Similarly, we have

$$\sum_{j=1}^{m-3}\binom{m}{j}\left\|\phi_m^\ell W_m^0(\partial_x^{j}v)\partial_y f_{m-j}\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\leq CA^{m-6}\left((m-5)!\right)^{3(1+\sigma)}.$$

Then the desired estimate (5.5) follows. (c) It remains to prove that

$$\sum_{j=1}^{m-3}\binom{m}{j}\left\|\phi_m^\ell W_m^0(\partial_x^{j}v)(\partial_y\partial_x f_{m-j})\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\leq CA^{m-6}\left((m-5)!\right)^{3(1+\sigma)}. \tag{5.10}$$

The proof is quite similar to the previous step. To do so we first write

$$\begin{aligned}&\sum_{j=1}^{m-3}\binom{m}{j}\left\|\phi_m^\ell W_m^0(\partial_x^{j}v)(\partial_y\partial_x f_{m-j})\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\ =&\sum_{j=1}^{\left[m/2\right]}\binom{m}{j}\left\|\phi_m^\ell W_m^0(\partial_x^{j}v)(\partial_y\partial_x f_{m-j})\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\ &+\sum_{j=\left[m/2\right]+1}^{m-3}\binom{m}{j}\left\|\phi_m^\ell W_m^0(\partial_x^{j}v)(\partial_y\partial_x f_{m-j})\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\ =&\ \tilde S_1+\tilde S_2.\end{aligned}$$

For the term $\tilde S_1$, we use

$$\phi_m^\ell\leq\phi_m^0\leq\phi_{j+2}^0\phi_{m-j+1}^0,\quad W_m^0\leq W_{m-j+1}^0 \text{ for } j\geq 2,$$

to obtain

$$\tilde S_1\leq\sum_{j=1}^{\left[m/2\right]}\binom{m}{j}\left\|\phi_{j+2}^0\partial_x^j v\right\|_{L^\infty\left([0,T]\times\mathbb{R}_+^2\right)}\left(\phi_{m-j+1}^0W_{m-j+1}^0\partial_y\partial_x f_{m-j}\right)_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}.$$

Then repeating the arguments used to estimate S_1 and S_2 in the previous step, we can deduce that

$$\tilde S_1\leq CA^{m-6}\left((m-6)!\right)^{3(1+\sigma)}.$$

As for $\tilde S_2$, using the inequalities

$$\phi_m^\ell\leq\phi_m^0\leq\phi_{j+1}^0\phi_{m-j+2}^0,\quad W_m^0\leq W_{m-j+2}^0 \text{ for } j\geq 2,$$

gives

$$\begin{aligned}\tilde S_2\leq&\sum_{j=\left[m/2\right]+1}^{m-3}\binom{m}{j}\left\|\phi_{j+1}^0\partial_x^j v\right\|_{L^\infty([0,T]\times\mathbb{R}_+;\ L^2(\mathbb{R}_x))}\\ &\qquad\times\left\|\phi_{m-j+2}^0W_{m-j+2}^0\partial_y\partial_x f_{m-j}\right\|_{L^2([0,T]\times\mathbb{R}_+;\ L^\infty(\mathbb{R}_x))}.\end{aligned}$$

Then repeating the arguments used to estimate S_2 in the previous step, we have

$$\tilde{S}_2 \leq CA^{m-6}\left((m-5)!\right)^{3(1+\sigma)}.$$

This along with the estimate on $\tilde{S}_1$ yields (5.10). Finally, combining (5.3), (5.4), (5.5) and (5.10) gives the desired estimate in Lemma 5.3, and thus we have that the proof is complete. ◻

LEMMA 5.4. *Under the same assumption as in Proposition* 2.3*, we have*

$$\begin{aligned}\sum_{j=1}^{m}\binom{m}{j}&\left\|\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell(\partial_x^j u)f_{m+1-j}\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\ &+\sum_{j=1}^{m-1}\binom{m}{j}\left\|\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell\left[\partial_y\left(\partial_y\omega/\omega\right)\right](\partial_x^j v)(\partial_x^{m-j}u)\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\ \leq m\,C&\left\|\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}+CA^{m-6}\left((m-5)!\right)^{3(1+\sigma)}.\end{aligned}$$

The proof of this lemma is quite similar to Lemma 5.3, so we omit it.

LEMMA 5.5. *Under the same assumption as in Proposition* 2.3*, we have*

$$2\left\|\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell\left[\partial_y\left((\partial_y\omega)/\omega\right)\right]f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\leq C\left\|\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}.$$

Proof. This is just a direct verification. Indeed, Lemma 3.1 gives

$$\begin{aligned}&\left\|\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell\left[\partial_y\left((\partial_y\omega)/\omega\right)\right]f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\ &\leq\left\|\left[\partial_y\left((\partial_y\omega)/\omega\right)\right]\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\ &\quad+\left\|\left[\partial_y\left((\partial_y\omega)/\omega\right),\ \Lambda_\delta^{-2}W_m^\ell\right]\phi_m^\ell f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}\\ &\leq C\left\|\Lambda_\delta^{-2}\phi_m^\ell W_m^\ell f_m\right\|_{L^2\left([0,T]\times\mathbb{R}_+^2\right)}.\end{aligned}$$

Then the desired estimate follows and thus the proof of Lemma 5.5 is complete. ◻

Proof of Proposition 5.2. In view of (5.2), we combine the estimates in Lemmas 5.3–5.5, to get the first estimate in Proposition 5.2. The second one can be treated quite similarly and the main difference is that additionally we will use here the inductive estimates on the terms of the following form,

$$\left\|\partial_y^2\Lambda^{-2/3}\phi_j^0 W_j^0 f_j\right\|_{L^2([0,T]\times\mathbb{R}_+^2)},\quad 6\leq j\leq m,$$

while in the proof of Lemma 5.3, we only used the estimates on the following two forms

$$\left\|\phi_j^0 W_j^0 f_j\right\|_{L^\infty([0,T];\ L^2(\mathbb{R}_+^2))},\quad\left\|\partial_y\phi_j^0 W_j^0 f_j\right\|_{L^2([0,T]\times\mathbb{R}_+^2)},\quad 6\leq j\leq m.$$

So we omit the treatment of the second estimate for brevity, and thus the proof of Proposition 5.2 is complete. ◻

Completeness of the proof of Proposition 2.3. The estimates (2.15) follow from the combination of Propostion 5.1 and the first estimate in Proposition 5.2, while the estimate (2.17) in Proposition 2.3 follows from Propostion 5.1 and the second estimate in Proposition 5.2. The treatment of (2.16) is exactly the same as (2.15). The proof of Proposition 2.3 is thus complete. ◻

6. Appendix. Here we deduce the equation fulfilled by f_m (cf. [21]). Recall that

$$f_m = \partial_x^m \omega - \frac{\partial_y \omega}{\omega} \partial_x^m u, \quad m \geq 1,$$

where u is a smooth solution to the Prandtl equation (1.1) and $\omega = \partial_y u$. We will verify that

$$\partial_t f_m + u\partial_x f_m + v\partial_y f_m - \partial_y^2 f_m = \mathcal{Z}_m, \tag{6.1}$$

where

$$\begin{aligned}\mathcal{Z}_m = &-\sum_{j=1}^{m} \binom{m}{j} (\partial_x^j u) f_{m+1-j} - \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j v)(\partial_y f_{m-j}) \\ &- \left[\partial_y \left(\frac{\partial_y \omega}{\omega}\right)\right] \sum_{j=1}^{m-1} \binom{m}{j} (\partial_x^j v)(\partial_x^{m-j} u) - 2\left[\partial_y \left(\frac{\partial_y \omega}{\omega}\right)\right] f_m.\end{aligned}$$

To do so, we first notice that

$$u_t + uu_x + vu_y - u_{yy} = 0 \tag{6.2}$$

and

$$\omega_t + u\omega_x + v\omega_y - \omega_{yy} = 0.$$

Thus by Leibniz's formula, $\partial_x^m u, \partial_x^m \omega$ satisfy, respectively, the following equations:

$$\begin{aligned}&\partial_t \partial_x^m u + u\partial_x \partial_x^m u + v\partial_y \partial_x^m u - \partial_y^2 \partial_x^m u \\ &= -\sum_{j=1}^{m} \binom{m}{j} (\partial_x^j u)(\partial_x^{m-j+1} u) - \sum_{j=1}^{m} (\partial_x^j v)(\partial_y \partial_x^{m-j} u) \\ &= -\sum_{j=1}^{m} \binom{m}{j} (\partial_x^j u)(\partial_x^{m-j+1} u) - \sum_{j=1}^{m-1} (\partial_x^j v)(\partial_y \partial_x^{m-j} u) - (\partial_x^m v)(\partial_y u)\end{aligned} \tag{6.3}$$

and

$$\begin{aligned}&\partial_t \partial_x^m \omega + u\partial_x \partial_x^m \omega + v\partial_y \partial_x^m \omega - \partial_y^2 \partial_x^m \omega \\ &= -\sum_{j=1}^{m} \binom{m}{j} (\partial_x^j u)(\partial_x^{m-j+1} \omega) - \sum_{j=1}^{m} (\partial_x^j v)(\partial_y \partial_x^{m-j} \omega) \\ &= -\sum_{j=1}^{m} \binom{m}{j} (\partial_x^j u)(\partial_x^{m-j+1} \omega) - \sum_{j=1}^{m-1} (\partial_x^j v)(\partial_y \partial_x^{m-j} \omega) - (\partial_x^m v)(\partial_y \omega).\end{aligned} \tag{6.4}$$

In order to eliminate the last terms on the right sides of the above two equations, we observe $\partial_y u = \omega > 0$ and thus multiply (6.3) by $-\frac{\partial_y \omega}{\omega}$, and then add the resulting equation to (6.4); this gives

$$\partial_t f_m + u\partial_x f_m + v\partial_y f_m - \partial_y^2 f_m = \mathcal{Z}_m,$$

where

$$
\begin{aligned}
\mathcal{Z}_m = &-\sum_{j=1}^{m}\binom{m}{j}(\partial_x^j u)f_{m+1-j} - \sum_{j=1}^{m-1}\binom{m}{j}(\partial_x^j v)(\partial_y f_{m-1}) \\
&-\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]\sum_{j=1}^{m-1}\binom{m}{j}(\partial_x^j v)(\partial_x^{m-j}u) + (\partial_x^m u)f_1 \\
&+\left(\partial_t\left(\frac{\partial_y\omega}{\omega}\right) + u\partial_x\left(\frac{\partial_y\omega}{\omega}\right) + v\partial_y\left(\frac{\partial_y\omega}{\omega}\right) - \partial_y^2\left(\frac{\partial_y\omega}{\omega}\right)\right)\partial_x^m u \\
&-2\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]\partial_y\partial_x^m u.
\end{aligned}
$$

On the other hand we notice that

$$
\begin{aligned}
&\partial_t\left(\frac{\partial_y\omega}{\omega}\right) + u\partial_x\left(\frac{\partial_y\omega}{\omega}\right) + v\partial_y\left(\frac{\partial_y\omega}{\omega}\right) - \partial_y^2\left(\frac{\partial_y\omega}{\omega}\right) \\
&= \frac{1}{\omega}\left(\partial_t\partial_y\omega + u\partial_x\partial_y\omega + v\partial_y\partial_y\omega - \partial_y^2\partial_y\omega\right) \\
&\quad -\frac{\partial_y\omega}{\omega^2}\left(\partial_t\omega + u\partial_x\omega + v\partial_y\omega - \partial_y^2\omega\right) + 2\frac{\partial_y\omega}{\omega}\partial_y\left(\frac{\partial_y\omega}{\omega}\right) \\
&= -\partial_x\omega + \frac{(\partial_x u)(\partial_y\omega)}{\omega} + 2\frac{\partial_y\omega}{\omega}\partial_y\left(\frac{\partial_y\omega}{\omega}\right).
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
\mathcal{Z}_m = &-\sum_{j=1}^{m}\binom{m}{j}(\partial_x^j u)f_{m+1-j} - \sum_{j=1}^{m-1}\binom{m}{j}(\partial_x^j v)(\partial_y f_{m-1}) \\
&-\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]\sum_{j=1}^{m-1}\binom{m}{j}(\partial_x^j v)(\partial_x^{m-j}u) + (\partial_x^m u)f_1 \\
&+\left(\partial_x\omega - \frac{(\partial_x u)(\partial_y\omega)}{\omega}\right)\partial_x^m u + 2\frac{\partial_y\omega}{\omega}\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\partial_x^m u - 2\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]\partial_y\partial_x^m u \\
= &-\sum_{j=1}^{m}\binom{m}{j}(\partial_x^j u)f_{m+1-j} - \sum_{j=1}^{m-1}\binom{m}{j}(\partial_x^j v)(\partial_y f_{m-1}) \\
&-\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]\sum_{j=1}^{m-1}\binom{m}{j}(\partial_x^j v)(\partial_x^{m-j}u) + \left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)^2\right]\partial_x^m u \\
&-2\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]\partial_x^m\omega \\
= &-\sum_{j=1}^{m}\binom{m}{j}(\partial_x^j u)f_{m+1-j} - \sum_{j=1}^{m-1}\binom{m}{j}(\partial_x^j v)(\partial_y f_{m-1}) \\
&-\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]\sum_{j=1}^{m-1}\binom{m}{j}(\partial_x^j v)(\partial_x^{m-j}u) - 2\left[\partial_y\left(\frac{\partial_y\omega}{\omega}\right)\right]f_m.
\end{aligned}
$$

Next we will give the boundary value of $\partial_y f_m$ and $\partial_t f_m - \partial_y^2 f_m$. In view of (6.2), we infer, recalling $u|_{y=0} = v|_{y=0} = 0$,

$$\partial_y \omega\big|_{y=0} = \partial_y^2 u\big|_{y=0} = 0.$$

As a result, observing

$$\partial_y f_m = \partial_y \partial_x^m \omega - \left[\partial_y \left(\frac{\partial_y \omega}{\omega}\right)\right] \partial_x^m u - \left(\frac{\partial_y \omega}{\omega}\right) \partial_y \partial_x^m u,$$

we have

$$\partial_y f_m|_{y=0} = 0. \tag{6.5}$$

Direct verification shows

$$\mathcal{Z}_m|_{y=0} = -2\left[\partial_y \left(\frac{\partial_y \omega}{\omega}\right)\right] f_m\Big|_{y=0},$$

and thus

$$\left(\partial_t f_m - \partial_y^2 f_m\right)|_{y=0} = \mathcal{Z}_m|_{y=0} = -2\left[\partial_y \left(\frac{\partial_y \omega}{\omega}\right)\right] f_m\Big|_{y=0}, \tag{6.6}$$

due to the equation fulfilled by f_m.

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